

MOVING ENVELOPES OF STARS

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★ ★ OF STARS ★ ★

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TRANSLATED BY SERGEI GAPOSCHKIN

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TRANSLATOR'S PREFACE

Several years ago I prepared a translation of this epoch-making discussion of *Moving Envelopes of Stars*. The demand for copies of this translation in mimeographed form was so great that Professor Donald H. Menzel suggested that it be made available in printed form.

I have raised with the author the question of including later research in the field. He agrees with me that this is unnecessary since the present work is complete in itself, and to include the results of later investigations would require rewriting the whole book. The fundamental relations given here remain true as they were when the author presented them.

I acknowledge with thanks the invaluable help of my wife in arriving at the precise meaning of the Russian text.

S. G.

INTRODUCTION

Stars with bright-line spectra are among the most interesting subjects in modern astrophysics. The spectra of such stars undergo large and rapid changes. During times comparable with the lives of human beings, or sometimes even before our eyes, the lines change their position and intensity, appear and disappear. These phenomena point to disturbed conditions in the stellar atmospheres and sharp transitions from one state to another.

Analysis of the spectrograms shows that during such times there is a powerful ejection of matter from the stars. In some cases this process is explosive; the envelopes of the star are thrown off with large velocities, and travel into interstellar space. These cases are the new stars, the supernovae and the novalike stars. A particularly large amount of matter is thrown out in the explosion of a supernova. The planetary nebulae which envelop hot stars are apparently the result of such explosions. In other cases the matter is ejected more or less continuously during a long interval. This process leads to the formation of stars with extensive atmospheres—Wolf Rayet stars, P Cygni stars, Be stars, and supergiants.

The physical phenomena that occur in the envelopes which are ejected by stars are of great interest to astrophysicists. The phenomena are interesting in themselves because of the extreme conditions that exist in the envelopes. Furthermore—and this is what is most important—the study of these phenomena allows us to penetrate more deeply into the nature of the stars.

A study of the illumination of the envelopes enables us to determine the energy radiated by the stars in different regions of the spectrum, the behavior of moving envelopes, and the forces exerted on the envelope by the star. All these, in the end, lead to an understanding of the causes that produce these gigantic cosmic catastrophes and raise the curtain on the structure of stellar nuclei and

the mysterious sources of their energy. It is especially significant that in observing these stars we are direct witnesses of phenomena that have tremendous cosmogonic significance.

However, we must point out the great gap between theory and observation in this field. While observational astrophysics has accumulated, is accumulating, and will actively accumulate very rich observational material, theoretical astrophysics cannot yet cope adequately with the interpretation of this material.

One of the chief needs of theoretical astrophysics at the present time is, in my opinion, the development of the theory of radiative equilibrium for a moving medium. The interpretation of phenomena that occur in moving atmospheres has hitherto been based either on the theory of radiative equilibrium for a stationary atmosphere or on the theory for gaseous nebulae (that is, for a transparent medium). It is clear that neither can lead to a satisfactory treatment. A few authors have made attempts to solve some problems connected with the theory of radiative equilibrium of a moving medium, but the treatment has not been fruitful.

The purpose of the present work is to give a theory of radiative equilibrium of a moving medium and to apply it to the moving envelopes of stars. Because of the breadth of the problem, only the basis of the theory and its most immediate applications are given here.

We first consider the main problem. The basic process occurring in the stellar atmosphere is the transfer of energy by radiation. The radiation flux from the interiors of the stars undergoes considerable changes on reaching the atmosphere (as a result of diffusion, fluorescence, transformation of the radiative energy into heat energy, and so forth). Part of it is reflected and part emerges to be observed in the form of a spectrum. The problem for the astrophysicist is therefore to deduce the structure of the atmosphere and the phenomena occurring in it, on the basis of the appearance of the spectra. Clearly this requires the solution of the reverse problem: for a given atmosphere and radiation flux, to determine the field of radiation in the atmosphere and, in particular, the spectral characteristics. In solving this problem we usually make the justifiable assumption that each volume element emits as much energy as it absorbs. We then say

that the atmosphere is in a state of radiative equilibrium; the solution of the problem involves the theory of radiative equilibrium.

The theory of radiative equilibrium is a most important province of astrophysics. For “ordinary” stars this theory has attained considerable success in the hands of Schwarzschild, Milne, Eddington, and others, although it still has limitations; for example, in the problem of the photosphere, it has usually been assumed that the absorption coefficient is independent of the frequency, and in constructing contours for absorption lines, atoms with two, or in the best cases three, levels have been considered. These limitations were chiefly caused by the mathematical complexity of the theory, which leads to nonlinear integral-differential equations.

The processes involved in energy transfer in the envelopes ejected by stars are very specific. Radiation passing from the star into the envelope has a very small density in comparison with that associated with the temperature which characterizes the relative distribution of energy in the spectrum. In such cases the interaction between radiation and matter leads to a reduction of this discrepancy. More precisely, the short-wavelength radiation arriving from the star is transformed into long-wavelength radiation of the envelope (partly into radiation in the visual region of the spectrum). Bright lines appear in the spectra of stars as a consequence of fluorescence of this type.

These processes are especially simple in planetary nebulae. The small density of matter (of the order of 10^{-20} gm cm $^{-3}$) and the small radiation density (the incident radiation is diminished by a factor of 10^{13} to 10^{15}) lead to a very low excitation, in consequence of which the nebulae appear transparent to the radiation of subordinate series. The process of fluorescence in the nebulae takes place as follows: ionization of atoms from the fundamental state first occurs under the action of radiation, the capture of electrons by ions follows, and then the atoms “cascade” from level to level, unaffected either by the radiation or by collisions. In such a case (if we consider hydrogen) we can apply the theory of radiative equilibrium only for L_c and L_α radiation. This process is so simple that the detailed development of the theory of the radiation of nebulae presents no difficulty. The chief successes in this field are connected with the

names of Zanstra (method for the determination of temperature of the central stars), Ambartsumian (the theory of radiative equilibrium for L_c and L_α radiation), and Cillié (theory of the Balmer decrement).

The treatment of envelopes of small radius is much more complicated. In this case, ionization occurs not only from the fundamental state but also from excited states, and the envelopes appear non-transparent to the radiation of the lines of subordinate series. It is clear that for such envelopes the results obtained for nebulae are not applicable. The theories of Zanstra and Cillié have been widely applied to the WR and Be stars, novae, and so forth. On the assumption of the transparency of these envelopes, the contours of emission lines have been computed, and many far-reaching conclusions have been derived. Other investigators have applied to these envelopes theories which are derived for the normal atmosphere (for example, the Boltzmann formula).

We should in fact develop a special theory of radiative equilibrium for moving envelopes. This theory should account for the fact that, in a medium which moves with a velocity gradient, the spectral lines exhibit the Doppler effect.

The presence of a velocity gradient leads to very serious theoretical difficulties. However, as will be shown later, there are very fundamental simplifications, chiefly connected with the fact that, owing to the presence of the velocity gradient, the line radiation reaches the observer not only from the external regions of the medium but also (on account of the Doppler effect) from the internal regions. Hence we can go much further in the theory of radiative equilibrium of a moving medium than in the theory of a stationary atmosphere. In particular, it is quite possible to construct a theory of polychromatic radiative equilibrium for a moving medium (that is, for atoms with a large number of levels).

In the present work we are not occupied with the general formal solution of the problem. We consider particular cases of the theory in order of increasing complexity. A definite class of observed objects corresponds to each of these cases. The astronomical application of the results obtained is thus simplified.

The plan of this work is as follows. In the first chapter we consider a medium in which the density of matter, the density of ionizing

radiation, and the velocity gradient are constant. For this model, the problem is reduced to a system of algebraic equations (not integral-differential, as in the general case). As a result we can determine the degree of excitation and ionization of real atoms in this medium and also the relative intensities of emission lines, the latter for the first time in a nontransparent medium.

In the second chapter we apply these results to the atmospheres of bright-line stars of early spectral class. We determine the degree of excitation and that of radiation in the lines as functions of distance from the center of the star. We also give the elementary theory of the contours of spectral lines for fundamental types of moving atmospheres.

In the following chapters we consider envelopes which have optical depths, beyond the fundamental series, greater than unity. In such envelopes the density of the ionizing radiation cannot be assumed. First (Chapter III) we consider that the absorption of light beyond the limits of subordinate series does not play any part; then (Chapter IV) this absorption is taken into account. The first case has application to planetary nebulae, the second, to the envelopes of new stars.

In these chapters we derive the equations of radiative transfer in the spectral lines by taking the Doppler effect into account, and give a solution of the equations of the theory of radiative equilibrium. As a result we find formulae which express the degree of excitation and ionization in the envelopes as functions of the optical depths. Here, as in the previous chapter, the important role played by the velocity gradient is apparent.

The results are applied to analysis of the physical characteristics of the objects under investigation. We give a method for the determination of the temperature of an envelope, and refine the method of Zanstra for the determination of the temperatures of the stars. We examine the question of radiation pressure in the envelope. In the last chapter we consider envelopes with optical depths that are very great in comparison with unity, not only beyond the limit of the fundamental series, but also beyond the limits of subordinate series. This is already essentially a photospheric problem. Further consideration leads to an elucidation of one of the most difficult

problems in astrophysics—the question of the origin and behavior of bright lines in spectra of stars of later classes.

The present work touches on the whole theory of an important astrophysical problem. The history of these questions and the present state of the theory will be found in textbooks of theoretical astrophysics and in numerous articles. Therefore, I present here only the results which I have obtained myself. These results have been partially published in the form of separate articles in the *Annals of the Academy of Sciences of the Soviet Union*, the *Russian Astronomical Journal*, and the *Scientific Memoirs of Leningrad State University*. With little modification, these investigations constitute the chief part of the present contribution.

CONTENTS

	PAGE
I. The Homogeneous Medium	3
1. Fundamental Equations	3
2. Degree of Excitation and Ionization	7
3. Relative Intensities of Lines	9
4. On the Transparency of the Medium to Line Radiation	11
5. Comparison with Observation	14
 II. Stars of the Early Classes with Bright Lines	 20
1. The Lower Boundary of the Atmosphere	21
2. Excitation Changes Along the Radius	26
3. Contours of Spectral Lines	29
 III. Gaseous Nebulae	 41
1. The Field of L_c Radiation	42
2. The Field of L_α Radiation	44
3. Temperatures of Nebulae and Nuclei	50
 IV. Envelopes of New Stars	 62
1. The Atom with Three States	63
2. The Real Atom	69
3. Collisions and General Absorption in the Envelope	72
4. Certain Applications	75

V. Stars of Late Class with Bright Lines	82
1. Origin of "Combination Spectra"	84
2. Optical Depth of the Envelope Beyond the Limits of Subordinate Series	91
3. General Considerations	94
Summary	99
References	103
Index	105

MOVING ENVELOPES OF STARS

I

THE HOMOGENEOUS MEDIUM

Consider a gaseous medium, placed at some distance from the star. Assume that the optical depth of this medium beyond the fundamental series of a given atom is less than 1, but that in the spectral lines it is greater than 1. The following physical processes occur in such a medium. The atoms are ionized by radiation coming from the star. After ionization, electrons are captured by the ions, and the resulting quanta in the continuous spectrum emerge from the medium without hindrance. Furthermore, as electrons pass from one state to another, quanta are radiated as spectral lines and leave the medium, in general, only after several scatterings.

Our problem is to find the distribution of atoms in different states at any point in the medium, and also to find the radiative field of the medium in all spectral lines and beyond the series limits, on the assumption of radiative equilibrium.

We first assume that the medium is stationary or that it moves without velocity gradient. In such a case the line quanta can leave the medium practically only in places close to the boundary. If the quantum occurs in the interior of the medium, it must penetrate the whole medium and undergo a large number of scatterings (of the order of the square of the optical thickness of the medium in the lines) before emerging. Clearly the quality of radiation at any point will depend very much on the conditions at other points, particularly at the boundaries (for example, the quality of the radiation will change if the medium is decreased or increased in dimensions). On account of all these conditions we see that the problem appears to be exceedingly difficult, and hitherto it has been solved only in a few of the simplest cases (medium plane-parallel, and atoms with only two discrete states).

We assume now that the medium has a velocity gradient. In this case the line quanta can escape from the interior of the medium,

because of the Doppler effect. Therefore, with a large velocity gradient, the situation at a given point sufficiently far from the boundaries will depend little on what happens at another point, and particularly at the boundaries. It is easy to see that the present case is much simpler than the foregoing one.

In the present chapter we shall consider an infinite homogeneous medium which is moving in such a way that the relative velocity of two points is proportional to the distance between them (we specify an infinite medium only in the interests of rigor; the results obtained here will apply to the internal regions of any medium that possesses the same properties). We assume the density of ionizing radiation to be constant and given. In such a case our problem is reduced to the solution of a system of algebraic equations. The solution of this system makes it possible to find the degree of excitation and ionization of real atoms (not the ideal with two or three states). Thus we shall derive the relative intensities of spectral lines radiated by the medium assumed.

In Sec. 1 of this chapter we derive the basic equations of the problem. In Sec. 2 these equations are solved for the hydrogen atom and the ionized helium atom. In Sec. 3 we compute relative intensities of the lines of these atoms. In Sec. 4 we discuss the question of the transparency of the medium to line radiation. Finally, in Sec. 5 we give theoretical line intensities and compare them with observation.

1. Fundamental Equations

Let n_i be the number of atoms in the i th state, n^+ the number of ionized atoms, and n_e the number of free electrons per cubic centimeter. The number of ionizations from the i th state per cubic centimeter per second will be denoted by $n_i B_{ic} \rho_{ic}$, where B_{ic} is the Einstein coefficient of absorption and ρ_{ic} is the radiation density beyond the i th series; that is, we denote the number of captures in the i th state by $n_e n^+ C_i(T_e)$, where $C_i(T_e)$ is some function of the electron temperature T_e . Finally, we denote the number of spontaneous transitions from the i th state to the k th state by $n_i A_{ik}$, where A_{ik} is the Einstein coefficient of spontaneous transition.

If the medium were stationary, then the transitions from the i th

to the k th state would be exactly balanced by transitions from the k th state to the i th, because all quanta radiated in spectral lines would have been absorbed in the same medium. When a velocity gradient exists, however, the number of transitions from the i th state to the k th is larger than the number of inverse transitions, because some fraction of the quanta in the corresponding line will leave the medium owing to the Doppler effect. We denote this fraction by β_{ki} . Then the excess of transitions of type $i \rightarrow k$ over the inverse transitions will be $n_i A_{ik} \beta_{ki}$.

The number of transitions of atoms from the i th state to all other states should, in a stationary condition, be equal to the number of transitions into the i th state. In consequence we have

$$n_i \left(\sum_{k=1}^{i-1} A_{ik} \beta_{ki} + B_{ic} \rho_{ic} \right) = \sum_{k=i+1}^{\infty} n_k A_{ki} \beta_{ik} + n_e n^+ C_i(T_e) \quad (1)$$

$$(i = 1, 2, 3, \dots).$$

In these equations the quantity ρ_{ic} is assumed to be known, and

$$\rho_{ic} = W \rho_{ic}^* \quad (2)$$

where ρ_{ic}^* is the radiation density beyond the limit of the i th series at the surface of the star, and W is the dilution coefficient.

First we must determine the quantity β_{ik} . In attacking this problem we assume that the coefficient of absorption and the coefficient of emission in a line of frequency ν_{ik} are different from zero and constant in the interval $\Delta\nu_{ik}$, which is equal to

$$\Delta\nu_{ik} = 2 \frac{u}{c} \nu_{ik}, \quad (3)$$

where u is the mean thermal velocity of atoms and c is the velocity of light, and that these coefficients are equal to zero outside the interval. For the absorption coefficient in the line we have:

$$\alpha_{ik} = \frac{n_i B_{ik}}{c \Delta\nu_{ik}} \left(1 - \frac{g_i n_k}{g_k n_i} \right) h \nu_{ik}. \quad (4)$$

When the radiation emerging from point A reaches point B , at the distance s from A , it will be diminished by $e^{\alpha_{ik}s}$ and will be displaced in frequency by the quantity

$$\nu'_{ik} - \nu_{ik} = \frac{\nu_{ik}}{c} \frac{dv}{ds} s, \quad (5)$$

where dv/ds is the velocity gradient in the medium. Consequently, the fraction of the emergent radiation from A that is absorbed will be

$$\int_0^\infty e^{-\alpha_{ik}s} \left(1 - \frac{\nu'_{ik} - \nu_{ik}}{2\Delta\nu_{ik}} \right) \alpha_{ik} ds = 1 - \frac{1}{2u} \frac{dv}{\alpha_{ik} ds}. \quad (6)$$

Thus for the quantity β_{ik} we find:

$$\beta_{ik} = \frac{1}{2u} \frac{dv}{\alpha_{ik} ds}. \quad (7)$$

The expression obtained for β_{ik} should be inserted in Eq. (1). But we easily see that:

$$A_{ki}\beta_{ik} = \frac{(g_2/g_1)n_1 - n_2}{(g_k/g_i)n_i - n_k} \left(\frac{\nu_{ik}}{\nu_{12}} \right)^3 A_{21}\beta_{12}. \quad (8)$$

Therefore, as a result of this substitution we find:

$$\begin{aligned} n_i \left[x \sum_{k=1}^{i-1} \frac{(g_2/g_1)n_1 - n_2}{(g_i/g_k)n_k - n_i} \left(\frac{\nu_{ki}}{\nu_{12}} \right)^3 + \frac{B_{ic}\rho_{ic}^*}{A_{21}} \right] \\ = x \sum_{k=i+1}^{\infty} n_k \frac{(g_2/g_1)n_1 - n_2}{(g_k/g_i)n_i - n_k} \left(\frac{\nu_{ik}}{\nu_{12}} \right)^3 + \frac{n_e n^+ C_i}{W A_{21}}, \end{aligned} \quad (9)$$

where

$$x = \beta_{12}/W. \quad (10)$$

This system of Eqs. (9) determines the degree of ionization and excitation in the medium completely; in other words, it determines n_i/n_1 and $n_e n^+/W n_1$. In addition to the parameter x , other parameters enter Eqs. (9): the temperature of the star (by means of $B_{ic}\rho_{ic}^*$) and the temperature of the medium (by means of C_i). An attractive circumstance is that we have obtained a parameter that represents the ratio of the velocity gradient to the coefficient of dilution, not

these quantities separately. This means, among other things, that the more a particular velocity gradient affects the excitation and ionization, the less is the coefficient of dilution.

We have obtained the equation for the case where the medium is not transparent to the radiation in all lines. We now assume that the medium is transparent to lines of all series beginning from the j 's. Then it is obvious that we should set in our equations

$$\beta_{ik} = 1 \quad (i = j, j + 1, j + 2, \dots). \quad (11)$$

We know that gaseous nebulosities are completely transparent to radiation in the lines of subordinate series, but not transparent to radiation in the lines of fundamental series. Setting, in Eq. (1), $\beta_{ik} = 1$ ($i = 2, 3, 4, \dots$), $\beta_{ik} = 0$, and neglecting ionization from excited states, we have:

$$n_i \sum_{k=2}^{i-1} A_{ik} = \sum_{k=i+1}^{\infty} n_k A_{ki} + n_e n^+ C_i. \quad (12)$$

The system of Eqs. (12) has been discussed by Cillié.¹

2. Degree of Excitation and Ionization

In order to solve Eqs. (9) we must know the functions C_i and $B_{ic}\rho_{ic}^*$ for a given atom. For a hydrogen atom this function has the form¹

$$C_i(T) = \frac{2^9 \pi^5}{(6\pi)^{\frac{3}{2}}} \frac{e^{10}}{m^2 c^3 h^3} \left(\frac{m}{kT} \right)^{\frac{3}{2}} \frac{1}{i^3} e^{h\nu_{ic}/kT} E_i \left(\frac{h\nu_{ic}}{kT} \right), \quad (13)$$

$$B_{ic}\rho_{ic}^* = 4\pi \int_{\nu_{ic}}^{\infty} \alpha_{i\nu} \frac{J_{\nu}^* d\nu}{h\nu}, \quad (14)$$

$$\alpha_{i\nu} = \frac{2^6 \pi^4}{3\sqrt{3}} \frac{me^{10}}{ch^6 i^5} \frac{1}{\nu^3}, \quad (15)$$

where m and e are the mass and charge of the electron, h is Planck's constant, and k is Boltzmann's constant.

It is easy to see that if Eqs. (9) are solved for hydrogen, the result obtained can be used for ionized helium. In fact, we know that the spectral terms of He^+ are four times those of hydrogen, the

coefficients of spontaneous transition of He^+ are 16 times those of hydrogen, and the statistical weights of the states coincide (see the work of Ambartsumian²). Therefore we come to the following conclusion: the numbers $n_i/n_e n^+$ for ionized helium will be eight times smaller than the corresponding numbers for hydrogen if we assume the temperature four times larger.

We have solved Eqs. (9) numerically for the hydrogen atom. We assume here that the temperature of the medium coincides with that of the star. If these temperatures are different, then the quantities $B_{ic}\rho_{ic}^*$ will be functions of T_* , and the quantities C_i will be functions of T_e . It is easy, however, to make applications to other cases. This possibility is related to the fact that the relative values of the quantities C_i change very little with change of T_e . Therefore the relative quantities n_i will also depend only slightly on T_e . For example, for $T_e = 10,000^\circ$, all n_i will be approximately twice the values for $T_e = 20,000^\circ$.

We have considered the following examples: $T = 20,000^\circ$, $x = 0, 0.01, 0.1, 1.0$; $T = 50,000^\circ$, $x = 0, 0.1, 1.0, 10$. The results are given in Tables 1, 2, and 3.

Table 1. Values of $10^{-20}(n_e n^+ / W n_1)$

T (deg)	x					
	0	0.01	0.1	1.0	10	∞
20,000	0.027	0.022	0.015	0.014	—	0.014
50,000	11.8	—	9.4	7.3	6.8	6.8

Table 2. Values of $100n_k/n_1$ ($T = 20,000^\circ$)

k	x			
	0	0.01	0.1	1.0
2	1.06	0.97	0.29	0.036
3	0.64	.33	.039	.00056
4	.60	.20	.0085	.000058
5	.61	.16	.0037	.000012
6	.65	.14	.0021	.000004

Table 3. Values of n_k/n_1 ($T = 50,000^\circ$)

k	x			
	0	0.1	1.0	10
2	0.30	0.18	0.046	0.0067
3	.26	.13	.019	.00095
4	.28	.12	.012	.00030
5	.31	.11	.009	.00015
6	.34	.10	.007	.00010

Table 1 shows that the degree of ionization depends only very slightly on the parameter x . We may assume that for any value of x the degree of ionization is determined by the usual ionization formula, with the factor W on the right-hand side. (This is a result of our assumption of the small optical depth of the medium beyond the limits of the fundamental series. If this optical depth is great, then, as we shall see in Chapter IV, the change of degree of ionization with optical depth will depend greatly on x .)

On the contrary, the degree of excitation, which is very little different from the Boltzmann value for $x = 0$, decreases rapidly with increasing x . This means that to find the degree of excitation in moving envelopes we cannot use the Boltzmann formula.

The decrease of excitation with increase of x (especially rapid for high states) leads to the consequence that at sufficiently large values of x the medium becomes transparent to radiation in the lines of given series. The results obtained in this paragraph are not applicable in such cases; they will be considered in detail in Sec. 4.

3. Relative Intensities of Lines

In the foregoing section we determined the number of atoms in each state. This makes it possible to find the amount of energy radiated by the medium in any line. Now we shall find the relative intensities of the Balmer lines (that is, the so-called Balmer decrement) and of some lines of ionized helium.

Because only the fraction β_{ik} of the total number of quanta is radiated during the transition $k \rightarrow i$, the amount of energy radiated

by unit volume per unit time in the frequency ν_{ik} , and emerging from the medium, is

$$E_{ki} = n_k A_{ki} \beta_{ik} h \nu_{ik}, \quad (16)$$

or, remembering Eq. (8),

$$E_{ki} = n_k \frac{(g_2/g_1)n_1 - n_2}{(g_k/g_i)n_i - n_k} \left(\frac{\nu_{ik}}{\nu_{12}} \right)^3 A_{21} \beta_{12} h \nu_{ik}. \quad (17)$$

Assuming the intensity of $H\beta$ to be unity, we obtain by means of Eq. (17) the relative intensities of the Balmer lines:

$$\frac{E_{k2}}{E_{42}} = \frac{4(n_2/n_4) - 1}{\frac{1}{4}k^2(n_2/n_k) - 1} \left(\frac{\nu_{2k}}{\nu_{24}} \right)^4. \quad (18)$$

The Balmer decrement, computed from Eq. (18) and on the basis of Tables 2 and 3, is given in Tables 4 and 5.

Table 4. Balmer decrement ($T = 20,000^\circ$)

Line	x			
	0	0.01	0.1	1.0
$H\alpha$	0.67	0.98	2.00	5.20
$H\beta$	1.00	1.00	1.00	1.00
$H\gamma$	0.97	0.79	0.44	0.21
$H\delta$.87	.58	.22	.06

Table 5. Balmer decrement ($T = 50,000^\circ$)

Line	x			
	0	0.1	1.0	10
$H\alpha$	0.62	0.72	1.05	1.80
$H\beta$	1.00	1.00	1.00	1.00
$H\gamma$	1.02	0.85	0.72	0.48
$H\delta$	0.93	.65	.48	.29

It is also interesting to find the intensity ratio of the Balmer continuum to the lines. Because the amount of energy radiated beyond the limit of the Balmer series per cubic centimeter per second is equal to

$$E_{c2} = 5.52 \times 10^{-23} n_e n^+ T^{-\frac{1}{2}}, \tag{19}$$

we find for the ratio E_{c2}/E_{k2} :

$$\frac{E_{c2}}{E_{k2}} = \frac{7.3 \times 10^{-21}}{x T^{\frac{1}{2}}} \cdot \frac{n_e n^+ \frac{1}{4} k^2 (n_2/n_k) - 1}{W n_1 \quad 4 - n_2/n_1} \left(\frac{\nu_{12}}{\nu_{2k}} \right)^4. \tag{20}$$

With the help of Tables 1 and 2, Eq. (20) shows that for $T = 20,000^\circ$ and $x = 0.1$ the intensity of the Balmer continuum is about four times the intensity of $H\alpha$.

It follows from the above that Table 2, which gives the degree of excitation of hydrogen for $T = 20,000^\circ$, gives at the same time the degree of excitation of ionized helium for $T = 80,000^\circ$. With the data of that table we have computed the relative intensities of the three following lines of He II, which have the greatest interest: $\lambda 4686$ (transition $4 \rightarrow 3$), $\lambda 5411$ (transition $7 \rightarrow 4$), and $\lambda 6563$ (transition $6 \rightarrow 4$). The results of the computation are given in Table 6.

Table 6. Relative intensities of ionized helium lines ($T = 80,000^\circ$)

Ratio	x			
	0	0.01	0.1	1.0
E_{43}/E_{74}	3.0	4.0	5.5	11
E_{64}/E_{74}	0.7	0.9	1.0	1.5

These relative intensities of lines will be compared with observation in Sec. 5.

4. On the Transparency of the Medium to Line Radiation

As has been shown, the excitation diminishes rapidly with increasing x , and for sufficiently large values of x the medium becomes transparent to radiation in lines of all series beginning with $j + 1$.

The condition for nontransparency of the medium in lines of the j -series is the fulfillment of the inequality $\beta_{jk} < 1$. We cannot say beforehand, however, from which series the medium will begin to be transparent, because the quantity β_{jk} depends on the unknown quantity n_j . It is therefore desirable to find a simple method of estimating n_j .

We now give an approximate method for solution of Eqs. (9), based on the circumstance that for large values of x the number n_j rapidly diminishes with increasing j .

Adding all of Eqs. (9) term by term, beginning with j , we obtain

$$\sum_{i=j}^{\infty} n_i \left[x \sum_{k=1}^{j-1} \frac{(g_2/g_1)n_1 - n_2}{(g_i/g_k)n_k - n_i} \left(\frac{\nu_{ki}}{\nu_{12}} \right)^3 + \frac{B_{ic}\rho_{ic}^*}{A_{21}} \right] = \frac{n_e n^+}{WA_{21}} \sum_{i=j}^{\infty} C_i. \quad (21)$$

Neglecting the Einstein negative absorption, and writing

$$4A_{21} \frac{g_k}{g_i} \left(\frac{\nu_{ki}}{\nu_{12}} \right)^3 = D_{ki}, \quad \sum_{i=j}^{\infty} C_i = S_j, \quad (22)$$

we find, instead of Eq. (21),

$$\sum_{i=j}^{\infty} n_i \left(x \sum_{k=1}^{j-1} D_{ki} \frac{n_i}{n_k} + B_{ic}\rho_{ic}^* \right) = \frac{n_e n^+}{W} S_j. \quad (23)$$

Taking into consideration that the numbers n_i and D_{ki} decrease with increasing i , we limit ourselves to summation over i with the first member. Then we have

$$n_j \left(x \sum_{k=1}^{j-1} D_{kj} \frac{n_1}{n_k} + B_{jc}\rho_{jc}^* \right) = \frac{n_e n^+}{W} S_j. \quad (24)$$

Thus we arrive at the recurrence formula, which gives n_j if n_1, n_2, \dots, n_{j-1} are known.

If $j = 1$, we can obtain instead a more precise formula, which follows from Eq. (24), namely,

$$n_1 B_{1c}^* + n_2 B_{2c}\rho_{2c}^* = \frac{n_e n^+}{W} S_1. \quad (25)$$

Then Eq. (25) and the second of Eqs. (24) give

$$\frac{n_2}{n_1} = \frac{(1-p)B_{1c}\rho_{1c}^*}{xA_{21} + pB_{2c}\rho_{2c}^*}, \quad (26)$$

where $p = C_1/S_1$. Taking the third of Eqs. (24) we obtain

$$\frac{n_3}{n_1} = \frac{S_3}{S_2} \frac{xA_{21} + B_{2c}\rho_{2c}^*}{xD_{13} + xD_{23}\frac{n_1}{n_2} + B_{3c}\rho_{3c}^*} \cdot \frac{n_2}{n_1}, \quad (27)$$

and so on.

If, while computing according to Eq. (24), we obtained values of n_j such that $\beta_{jk} > 1$, it would mean that the medium is transparent to radiation of the j -series. Assuming that we have $\beta_{3k} > 1$, but $\beta_{2k} < 1$, this means that the medium is nontransparent to radiation of the Balmer series but transparent to radiation of the subsequent series. Remembering Eq. (8), we find that in such cases

$$\frac{A_{43}}{D_{34}} \frac{n_3}{n_1} < \beta_{12} < \frac{A_{32}}{D_{23}} \frac{n_2}{n_1}, \quad (28)$$

or

$$30 \frac{n_3}{n_1} < \beta_{12} < 8 \frac{n_2}{n_1}. \quad (29)$$

Computing, for example, the ratios n_2/n_1 and n_3/n_1 by Eqs. (26) and (27), with $T = 20,000^\circ$, we find that for $x = 1.0$

$$2 \times 10^{-4} < \beta_{12} < 3 \times 10^{-3}, \quad (30)$$

and for $x = 10$,

$$2 \times 10^{-6} < \beta_{12} < 3 \times 10^{-4}. \quad (31)$$

These are the conditions which must be fulfilled if the medium is nontransparent to the radiation of the Lyman and Balmer series, and transparent to the radiation of other series.

It is of interest to find the Balmer decrement for the case just

mentioned. To fulfill the last condition we should set $\beta_{jk} = 1$ ($j = 3, 4, 5, \dots$). The system of Eqs. (9) then takes the form

$$\left. \begin{aligned} n_1 B_{1c} W \rho_{1c}^* &= \beta_{12} \sum_{k=2}^{\infty} n_k D_{1k} + n_e n^+ C_1, \\ n_2 (A_{21} \beta_{12} + B_{2c} W \rho_{2c}^*) &= \frac{n_1}{n_2} \beta_{12} \sum_{k=3}^{\infty} n_k D_{2k} + n_e n^+ C_2, \\ n_1 \left[\beta_{12} \left(D_{1i} + \frac{n_1}{n_2} D_{2i} \right) + \sum_{k=3}^{i-1} A_{ik} + B_{ic} W \rho_{ic}^* \right] \\ &= \sum_{k=i+1}^{\infty} n_k A_{ki} + n_e n^+ C_i \quad (i = 3, 4, 5, \dots) \end{aligned} \right\} \quad (32)$$

We have solved Eqs. (32) numerically for the following two cases:

- I. $T = 20,000^\circ$, $\beta_{12} = 10^{-3}$, $W = 10^{-3}$, $x = 1.0$
- II. $T = 20,000^\circ$, $\beta_{12} = 10^{-5}$, $W = 10^{-6}$, $x = 10$.

The Balmer decrement is shown in Table 7.

Table 7. Balmer decrement for $\beta_{jk} = 1$ ($j = 3, 4, 5, \dots$)

Line	I	II
$H\alpha$	2.0	8.9
$H\beta$	1.0	1.0
$H\gamma$	0.80	0.91
$H\delta$.61	.84

The table shows that the intensities of $H\beta$, $H\gamma$, and $H\delta$ are nearly equal, but that the ratio of intensities $H\alpha/H\beta$ is very large. This Balmer decrement differs sharply from the one obtained earlier (see Tables 4 and 5). In reality, because the envelopes are of different nature, a combination of both Balmer decrements is possible.

In the next section we shall see that the Balmer decrement given in Table 7 is actually observed for some envelopes.

5. Comparison with Observation

There are many observational papers which contain data on the Balmer decrement in spectra of stars with bright lines. Usually the

observer compares his results with the Balmer decrements computed by Cillié.¹ He forgets, however, that the computations of Cillié refer only to envelopes which are transparent to radiation of lines of subordinate series. Because envelopes of small radius (that is, those of novae and novalike stars in the early stages, WR, P Cygni, and Be stars, and so forth) do not conform to this condition, the comparisons lack foundation.

It is no wonder, therefore, that there appears as a rule to be a large discrepancy between the theory of Cillié and the observations. The Balmer decrement computed by Cillié depends only on one parameter (the temperature) and for all legitimate values of this parameter it is practically constant ($H\alpha/H\beta = 3.0$, $H\gamma/H\beta = 0.5$). The observations show, however, a considerable variation in the Balmer decrement.

The observations should, in fact, necessarily be compared with a theory constructed for a nontransparent envelope. We now compare observation with the results obtained above.

(a) *P Cygni and Be Stars and Novae.* Table 8 contains values of the Balmer decrement determined for P Cygni by Beals,³ for stars of B0e and B3e by Karpov,⁴ (in the table we give the mean value for six stars), and for Nova Herculis by Greaves and Martin⁵ (in the table we give mean values for the first three months after the outburst). The temperatures of all these objects can be taken as close to 20,000°.

A comparison of Tables 4 and 7 shows that for a value of x near 0.1 there is satisfactory agreement between theory and observation.

Table 8. Observed Balmer decrement

Type	P Cygni	Be	N. Herc. 1934
$H\alpha$	2.45	2.25	1.90
$H\beta$	1.00	1.00	1.00
$H\gamma$	0.52	0.47	—
$H\delta$	—	.33	0.31

For the stars of type Be given in Table 8, Karpov has also determined the ratio of intensity of the Balmer continuum to the intensity

of $H\alpha$; he found that this ratio is 5.6 on the average, and established the great divergence between observation and the theory of Cillié (according to Cillié the ratio should be about 1.0). Since we deduced in the preceding section that if $x = 0.1$, the Balmer continuum is four times as bright as $H\alpha$, we are again convinced of satisfactory agreement between our results and observation.

(b) *Wolf-Rayet Stars*. It is well known that broad emission bands of hydrogen and ionized helium, superimposed upon each other, appear in the spectra of WR stars. In order to compare theory with observation it is necessary to separate these bands. For the star HD 192163 Beals³ made such a separation, assuming the Balmer decrement given by the theory of Cillié. He thus obtained a very unusual distribution of intensity among the lines of the Pickering series (with a maximum intensity for the line at 4861). If we assume that the Pickering decrement is "normal," we obtain approximately identical intensity for $H\alpha$, $H\beta$, and $H\gamma$.

The results of such a separation are given in Table 9.

Table 9. Intensities of lines in the star HD 192163

λ	$T' = 70,000^\circ$			$T' = 15,000^\circ$		
	H + He ⁺	H	He ⁺	H + He ⁺	H	He ⁺
6563	34	5	29	44	9	35
5711	28	—	28	31	—	31
4861	34	8	26	34	7	27
4541	24	—	24	23	—	23
4340	23	7	16	20	5	15
3923	6	—	6	5	—	5

In order to pass from the equivalent widths obtained from observation to the relative line intensities, it is necessary to adopt a law of energy distribution in the continuum of the stellar spectrum. Beals assumed that this distribution corresponds to $T' = 70,000^\circ$. From the theory of extended photospheres it follows, however, that this temperature should be much lower; in accordance with the recent determination of Vorontsov-Velyaminov,⁶ it is only $15,000^\circ$. Table 9 is accordingly made up for these two values of T' .

We may again regard such a Balmer decrement as compatible with our computations, for in high-temperature stars with very large range in the parameter x , the intensities of $H\alpha$, $H\beta$, and $H\gamma$ are close to one another (Table 5).

For the same star, HD 192163, according to Beals,³ the relative line intensities for He^+ are $E_{43}/E_{74} \approx 5$ and $E_{64}/E_{74} \approx 1$. From Table 6 we find that for $x = 0.1$ these values agree satisfactorily with the results of computation. We note that for the other stars of WR type studied by Beals, the ratio E_{43}/E_{74} is also of the order of a few units. If we assume, however, that the envelope is transparent to the radiation of lines of subordinate series, we obtain $E_{43}/E_{74} = 15$ and $E_{64}/E_{74} = 1.6$. These values are in sharp contradiction with the observations.

One should note that the investigators of WR stars usually (for example, when studying the contours of emission lines) make the hypothesis that the envelopes of these stars are transparent to line-radiation. This they justify on the basis of the weakness or absence of absorption lines. Our results on WR stars indicate that this hypothesis should be abandoned. In reality the emission lines seem to “fill in” the absorption lines (see Chapter II, Sec. 3).

We may also remark that, because of the superimposition of emission bands of H and He^+ on one another, it follows that in a rigorous consideration of band intensity the system of Eqs. (9) should be solved simultaneously for H and He^+ .

(c) *Long-Period Variables.* It is well known that bright Balmer lines are observed near maximum light for long-period variables. Estimation of the intensities of these lines, taken from the book of Merrill,⁷ are given in Table 10.

Table 10. Balmer decrement in spectra of long-period variables

Type	Me	Se	Ne
$H\alpha$	2	15	10
$H\beta$	2	12	10
$H\gamma$	20	5	5
$H\delta$	30	3	2

We see that in spectra of Me stars the Balmer decrement appears to be highly abnormal. This fact was long an enigma for astrophysicists. Shajn⁸ in 1935, however, convincingly showed that the reason for this anomaly of the Balmer decrement appears to be absorption by TiO. Later, Ambartsumian and Vashakidze,⁹ confirming the conclusion of Shajn, showed theoretically that no method whatever for the excitation of hydrogen atoms can lead to the observed Balmer decrement in Me spectra, if all the radiation of the hydrogen atoms reaches the observer. We should consider, however, that the envelopes of long-period variables are nontransparent to radiation of lines of subordinate series. Therefore the conclusion of Ambartsumian and Vashakidze needs to be generalized. Incidentally, our results also confirm the conclusion of Shajn (although a small inequality in the sense $H\alpha < H\beta < H\gamma$ appears to be realized).

Concerning the Balmer decrement in spectra of Se and Ne stars, from which the bands of TiO are absent, we see again that the observed decrement does not agree with the computation of Cillié, but agrees with our computations. We remark that in making this comparison, we presuppose that the Balmer emission in the spectra of long-period variables is produced by photoionization and recombination. Such a view is not commonly accepted. However, in Chapter V we shall advance weighty considerations in favor of it.

(d) *New Stars in Later Stages.* In the spectra of some new stars, several months after maximum, the following peculiar Balmer decrement has been observed: for more or less normal ratios of intensities $H\gamma/H\beta$ and $H\delta/H\beta$, the ratio of intensities $H\alpha/H\beta$ was very large. For example, according to Popper,¹⁰ in the spectrum of Nova Lacertae 1936 the latter ratio is 5 or 6, and according to Sayer,¹¹ in the spectrum of RS Ophiuchi 1933, it reached 10 and 12. The explanation of such phenomena was given at the end of the preceding paragraph: during this period, the envelopes of the stars (or parts of the envelopes) were nontransparent to radiation in the lines of the first two series, and transparent to radiation in the lines of subsequent series.

Summarizing this section, we recall that in computing intensities of bright lines we made the following two assumptions. First, the envelopes are nontransparent to radiation in lines of all (or some

specified first) series, and second, that the envelopes move with velocity gradients. Nontransparency of the envelopes is an observed fact; the majority of bright lines have absorption components. It should be emphasized that the motion of envelopes with a velocity gradient is also an observed fact. Spectrograms definitely indicate the motion of envelopes, and if this is so, motion cannot occur without velocity gradients. Independently of whether there is a velocity gradient along the radius, there must exist a velocity gradient in other directions owing to the curvature of the layers.

It is not difficult to estimate the value of the velocity gradient and also the value of the parameter x that enters into our equations. If the envelope is formed by matter ejected from the star, we have $dv/ds \approx v/r$, where v is the velocity of ejection and r the distance from the center of the star. For the value of x we have

$$x = \frac{10^{-5}}{\alpha_{1c} r} \frac{v}{u} \frac{1}{W}. \quad (33)$$

Because we can assume $\alpha_{1c} r \approx 1$, $v/u \approx 10$, $W \approx 10^{-3}$, Eq. (33) gives $x \approx 0.1$. This value coincides in order of magnitude with that assumed previously for the interpretation of the observed data.

II

STARS OF THE EARLY CLASSES WITH BRIGHT LINES

In the present chapter we shall consider the atmospheres of stars of types Wolf-Rayet, P Cygni, and Be. It is well known that in the spectra of these stars there are bright lines of hydrogen, helium, ionized helium, and other atoms with a very high ionization potential. In the spectra of WR and P Cygni stars the bright lines have as a rule dark components on the violet side, and in the spectra of stars of type Be the bright lines are superimposed on broad absorption bands. In accordance with the usual view, an ejection of matter takes place from the above-mentioned stars, an ejection which produces very extensive atmospheres. Bright lines are formed in these atmospheres as a result of photoionization and recombination. Judging from the width of the bright lines (more accurately speaking, bands), the velocity of ejection of matter for WR stars is of the order of 1000 km/sec, for P Cygni stars of about 100 km/sec. In order to explain the peculiarities of the line contours in the spectra of type Be one assumes that these stars have very rapid rotations (with a velocity of the order of several hundred km/sec).

It is usually assumed that the dark lines occur in a very thin reversing layer, with the bright lines in the extended transparent envelope lying above the reversing layer. It is more likely, however, that bright and dark lines are formed in the same extended non-transparent (for line radiation) envelope. We shall consider the latter view, and in this chapter we shall apply our previous results (Chapter I) to extended atmospheres.

In the previous chapter we discussed the excitation of atoms in a moving medium and the intensities of bright lines produced in this same medium. We assumed that the density of matter, the density of ionized radiation, and the velocity gradient do not change in the medium. However, in the real envelopes of stars all these quantities

are changing relative to the center of the star. We will, nevertheless, apply our formulae to real envelopes in the belief that they are applicable for each place separately while we change the parameter correspondingly. The reason for this is that in moving envelopes light quanta which are radiated in the spectral lines after ionization and recombination (after suffering a relatively small number of scatterings in a small region) will leave the envelope owing to the Doppler effect. In this small region the conditions indicated above may be fulfilled. (For details, see Chapter IV.)

In this chapter we consider the following problems. In Sec. 1 we find the lower limit of the atmosphere and some quantities which characterize the atmosphere as a whole. In Sec. 2 we clarify how the degree of excitation and the amount of energy which is radiated in the spectral lines change as functions of the distance of the star center. In the last section we discuss the question of spectral lines whose contours are formed in the moving atmospheres.

1. The Lower Boundary of the Atmosphere

In order to apply the results obtained earlier to a stellar atmosphere, it is necessary to know the distribution of densities and velocities in the atmosphere. In the first two paragraphs of this section, we consider the atmosphere, formed by matter ejected from the star, to have constant velocity. Then the density of matter will be inversely proportional to the square of the radius. Consideration of such a model is a very simple example of the application of our method; the development of such a method is our chief and immediate aim. On the other hand, this model corresponds very closely to the atmospheres of the stars of type WR and P Cygni, and also to a lesser degree to the atmospheres of stars of type Be. Therefore, the results obtained by means of this model will sufficiently characterize (at least qualitatively) the atmospheres of the above-mentioned stars.

We will first determine the lower boundary of the stellar atmosphere. For this determination it is necessary to know how the absorption in the continuum occurs in the upper layers of the star. It is easy to show that for the stars under consideration this absorption is caused not by photoionization and free-free transitions but by the scattering of light by free electrons. If this is so, then the

radius of the lower boundary of the atmosphere is determined by the relation

$$\int_{r_0}^{\infty} s_0 n_e dr = \frac{1}{3}, \quad (1)$$

where s_0 is the scattering coefficient of the free electron.

We have already agreed to assume that the density of the atmosphere falls off in inverse proportion to the square of the radius. Because in the atmospheres of hot stars the majority of atoms are in their ionized state, we find that

$$n_e = n_e^0 (r_0/r)^2, \quad (2)$$

where n_e^0 is the number of the free electrons at the lower boundary of the atmosphere. Substituting Eq. (2) in Eq. (1) we find:

$$n_e^0 r_0 = 0.5 \times 10^{24}. \quad (3)$$

This is the relation between the quantities r_0 and n_e^0 when electron scattering plays an important part in the continuous absorption in the upper parts of the star.

As has been said, this is easy to prove. To do this we will find the optical depth of the atmosphere beyond the Lyman limit. We have:

$$\tau_{1c} = 0.5 \times 10^{-17} \int_{r_0}^{\infty} n_1 dr = 0.5 \times 10^{-17} \frac{W n_1}{n_e n^+} 4 n_e^0 r_0. \quad (4)$$

If we assume that the radius of the star is ten times the radius of the sun ($r_0 = 7 \cdot 10^{11}$), then from Eq. (3) we find that $n_e^0 = 7 \cdot 10^{11}$. With these quantities we get from Eq. (4) $\tau_{1c} < 1$, if $T_* > 20,000^\circ$. For such small values of τ_{1c} , the optical depth of the atmosphere in the visible part of the spectrum, which is conditioned by the photo-ionization and free-free transitions, will be of the order of 0.001. Thus we have proved that the determination of the radius of the lower boundary of the atmosphere by formula (1) is justified.

While obtaining relation (4) we assumed that the usual ionization formula

$$n_e \frac{n^+}{n_1} = W f(T_*) \quad (5)$$

is justified in the atmosphere of the star, where

$$f(T_*) = \frac{(2\pi mkT_*)^{\frac{3}{2}}}{h^3} e^{-h\nu_{1c}kT_*}, \quad (6)$$

$$W = \frac{1}{4} \left(\frac{r_0}{r} \right)^2. \quad (7)$$

Because for the hottest stars $\tau_{1c} < 1$, the justification of this formula does not involve any doubt. Otherwise, it would be necessary to take into account the diminishing of the radiation from the star owing to extinction and to the presence of diffused radiation of the atmosphere beyond the Lyman series.

In the future we will assume always $\tau_{1c} < 1$. As a consequence it will follow not only that the ionization formula (5) is correct, but that the computational results of the foregoing chapter will also be correct; the derivation of the latter was based on the assumption that ionization is produced by the radiation coming directly from the star.

It is easy to obtain another relation between the quantities r_0 and n_e^0 . For this we only need to determine the amount of energy radiated by the atmosphere in any spectral line. Availing ourselves of the computational results obtained earlier for the hydrogen atom, we will talk in this chapter only about the radiation of the atmosphere in the Balmer lines. However, one should note that similar computations are easy to make for any atom.

Let H_k be the total energy radiated by the atmosphere in the k th line of the Balmer series. We have

$$H_k = \int n_k A_{k2} h\nu_{2k} \beta_{2k} dV, \quad (8)$$

where β_{2k} is the number of quanta which are coming out of the atmosphere owing to the Doppler effect; the integration is carried out over the whole volume of the atmosphere. This formula can be transformed into the following one:

$$H_k = \frac{16}{k^2} \left(\frac{\nu_{2k}}{\nu_{12}} \right)^3 A_{21} h\nu_{2k} \int f_k n_e n^+ dV, \quad (9)$$

where

$$f_k = \frac{n_k}{n_2} \frac{1 - (g_1/g_2) \cdot (n_2/n_1)}{1 - (g_2/g_k) \cdot (n_k/n_2)} \cdot \frac{Wn_1}{n_e n^+} x \cong \frac{n_k}{n_2} \cdot \frac{Wn_1}{n_e n^+} x \quad (10)$$

But the quantity f_k depends very little on the parameter x . For example, for $T = 20,000^\circ$, we have for $x = 0.01, 0.10$, and 1.00 , $10^{20}f_4 = 0.11, 0.19$, and 0.12 , respectively. Therefore the quantity f_k can be taken outside the integral sign. Assuming that $n_e = n^+$, we get instead of Eq. (9),

$$H_k = \frac{64\pi}{k^2} \left(\frac{\nu_{2k}}{\nu_{12}} \right)^3 A_{21} h \nu_{2k} f_k n_e^{02} r_0^3. \quad (11)$$

This is the final formula which determines the quantity H_k .

On the other hand, the same quantity H_k can be expressed by Zanstra's quantity A_k , which one obtains by the comparison of the intensity of a line and the intensity of the neighboring region of the continuous spectrum of the star:

$$H_k = A_k \frac{8\pi^2 r_0^2 h \nu_{2k}^4}{c^2} \cdot \frac{1}{e^{h\nu_{2k}/kT} - 1}. \quad (12)$$

The last two formulae give

$$\frac{8}{k^2} A_{21} f_k n_e^{02} r_0 = A_k \frac{\pi}{c^2} \frac{\nu_{12}^3}{e^{h\nu_{2k}/kT} - 1}. \quad (13)$$

Thus we have obtained a new relation [in addition to Eq. (3)], connecting the quantities r_0 and n_e^0 .

Equations (3) and (13) make it possible to determine the quantities r_0 and n_e^0 , that is, the radius of the lower boundary of the atmosphere and the electron density at this boundary. In the second of these formulae, in addition to the quantities r_0 and n_e^0 we find Zanstra's quantity A_k which is obtained from the observations and the temperature of the star. We will assume that the temperature of the star is known and consider two cases: (1) $T = 20,000^\circ$ (stars of types P Cygni and Be) and (2) $T = 50,000^\circ$ (WR stars).

In the first case we may take $f_4 = 0.15 \cdot 10^{-20}$ and $A_4 = 0.004$ (the mean value for many Be stars according to Mohler¹), and in the second $f_4 = 0.4 \cdot 10^{-21}$ and $A_4 = 0.002$ (for the star HD 192163 according to Beals). Equations (3) and (13) give for these two cases:

(1) $n_e^0 = 3.6 \cdot 10^{11}$, $r_0 = 1.4 \cdot 10^{12} = 20r_\odot$: (2) $n_e^0 = 1.5 \cdot 10^{12}$, $r_0 = 3.3 \cdot 10^{11} = 5r_\odot$.

These values for r_0 and n_e^0 seem to us very probable. For comparison we can for example note that for two eclipsing binaries of WR type found by Gaposchkin² the quantities are $r_0 = 5.8r_\odot$ and $r_0 = 5.5r_\odot$.

Using the quantities r_0 and n_e^0 we can, among other things, compute the amount of matter ejected by the star during a year. The ejected mass is obviously equal to

$$M = 4\pi r_0^2 n_e^0 m_H v \cdot 3.16 \times 10^7, \quad (14)$$

where m_H is the mass of the hydrogen atom and v is the velocity of ejection. This formula gives $M = 10^{-5}M_\odot$ for stars of WR and P Cygni type and $M = 10^{-6}M_\odot$ for stars of type Be.

We should note that the temperature of the stars is the weakest point. It is known that temperatures of stars of WR type, determined by the method of Zanstra, appear to be higher when the ionization potential of the element for which the temperature has been determined is greater. For example, the temperatures for the hydrogen lines, for ionized helium, and for four-times ionized oxygen are of the order of 20,000°, 60,000°, and 100,000° respectively. To a great extent this discrepancy is caused by the deviation of the radiation of the stars from the blackbody radiation, generally explained by the extended atmosphere. We can point out other factors which determine the temperature by the method of Zanstra. One of these factors is that, at the high stellar temperature, the optical depth of the atmosphere beyond the limit of the fundamental series of the atom should be smaller than 1. Consequently, the atmosphere will absorb only a small part of the radiation of the star beyond the limit of the fundamental series of the given atom, and the temperature determined from the line of this atom will be too small. As is seen from formula (4), this is the case with hydrogen if the temperature of the star is greater than 20,000°. For this case, we can refine the method of Zanstra by means of formula (4). We must multiply the left-hand side of Zanstra's equation by the quantity τ_{1c} . We take τ_{1c} from formula (4). Such a relation will depend very little on the temperature of the star, for with the increase of temperature the

number of quanta beyond the limit of the fundamental series increases but the quantity τ_{1c} decreases. In addition, this relation has a new unknown quantity—the radius r_0 of the star [the quantity n_e^0 can be excluded by means of Eq. (3)]. A more precise relation between T_* and r_0 is given in formula (13) if we take account of the nontransparency of the atmosphere for radiation in the lines beyond the limits of subordinate series.

2. Excitation Changes Along the Radius

As has been clarified in the previous chapter, the degree of excitation of atoms in a moving atmosphere depends on two parameters: temperature T_* and x . For x we find

$$x = \frac{1}{2un_1\kappa_{12}W} \left\langle \frac{dv}{ds} \right\rangle, \quad (15)$$

where u is the mean thermal velocity of the atoms, κ_{12} is the mean absorption coefficient in the spectral line for one atom, and $\langle dv/ds \rangle$ is the velocity gradient, averaged in direction. In order to find how the degree of excitation changes along a radius we have to express the quantity x as a function of r .

It is easy to see that in an atmosphere which is formed by the ejection of the matter from the star this velocity gradient is equal to

$$\frac{dv}{ds} = \frac{dv}{dr} \cos^2 \theta + \frac{v}{r} \sin^2 \theta, \quad (16)$$

where dv/dr is the velocity gradient along the radius and θ is the angle between the radius vector and a given direction. If the velocity of ejection is constant, then

$$\left\langle \frac{dv}{ds} \right\rangle = \frac{2}{3} \frac{v}{r}. \quad (17)$$

Using Eqs. (17), (5) and (2) in (15), we find

$$x = x_0 \left(\frac{r}{r_0} \right)^3, \quad (18)$$

where

$$x_0 = \frac{vf(T_*)}{3u\kappa_{12}r_0n_e^{02}}. \quad (19)$$

Because $n_e^0 r_0$ is equal to $0.5 \cdot 10^{24}$, and $u\kappa_{12} = 6.8 \cdot 10^{-8}$ for L_α , we have for hydrogen

$$x_0 = 10^{-17} \frac{vf(T_*)}{n_e^0}. \quad (20)$$

For a velocity of 100 km/sec and for the two above-mentioned values of temperature, Eq. (20) gives $x_0 = 0.001$ and 0.1.

We see that if the matter is ejected with constant velocity, the value of x increases proportionally to the cube of the radius. Consequently, the degree of excitation decreases very rapidly with increase of the radius. If we wish to represent the degree of excitation as a function of r , we must compute x by formula (18) and then we can use Tables 2 and 3 of the preceding chapter. We note that such a representation will be true only for the internal parts of the atmosphere. In the outer parts of the atmosphere, where it is transparent to the radiation of the lines in subordinate series, the quantities n_i/n_1 will be simply proportional to $n_e n^+/n_1$, that is, the degree of excitation will decrease as $1/r^2$.

It is interesting to find the boundary between the nontransparent and the transparent parts of the atmosphere, in other words, to find the upper boundary of the reversing layer. Because this boundary is different for different lines, we find it for the line $H\alpha$ as a definite case. We determine it by

$$\int_{r_1}^{\infty} n_2 \kappa_{23} dr = \frac{1}{3}. \quad (21)$$

For large values of x , we have approximately

$$\frac{n_2}{n_1} = (1 - p) \frac{B_{1c} \rho_{1c}^*}{x A_{21}} \quad (22)$$

[see formula (25) of the preceding chapter] and using Eqs. (5), (18), and (21) we have

$$\frac{r_1}{r_0} = \left[3(1 - p) \kappa_{23} \frac{B_{1c} \rho_{1c}^*}{x_0 A_{21}} \cdot \frac{n_e^{02} r_0}{f(T_*)} \right]^{\frac{1}{4}}. \quad (23)$$

For the last two cases formula (23) gives 12 and 4, respectively. Thus we see that these stars possess a very extended reversing layer. This result, among other things, should be important in the

computation of contours of absorption lines which are formed in the atmospheres of these stars.

The question of the amount of energy radiated by different layers of the atmosphere is inextricably bound up with the question of changes of the degree of excitation along the radius. If the degree of excitation is known as a function of the radius, this question is solved by means of formulae (8) and (9) (in which the integration must be carried only over the layer under consideration, and not over the whole volume of the atmosphere). Thus we can see the following important circumstances:

(1) Let H_k^0 and H_k' be the amount of energy radiated in the k th line of the Balmer series, corresponding to the reversing layer and the transparent part of the atmosphere respectively. Obviously we have

$$H_k^0 = 4\pi A_{k2} h\nu_{2k} \int_{r_0}^{r_1} n_k \beta_{2k} r^2 dr, \quad (24)$$

$$H_k' = 4\pi A_{k2} h\nu_{2k} \int_{r_1}^{\infty} n_k r^2 dr. \quad (25)$$

Evaluation of these integrals shows that the chief part of the energy is radiated not from the transparent part of the atmosphere but from the reversing layer. For example, in the first case ($T = 20,000^\circ$) the amount of energy radiated by the transparent part of the atmosphere constitutes only about 20 per cent. This conclusion contradicts the traditional opinion and proves the correctness of our point of view.

(2) Let $dH_k(r)/dr$ be the amount of energy radiated in the k th Balmer line by a spherical layer of unit thickness which is situated at distance r from the center of the star. The computations show that for different lines this quantity reaches a maximum at different places. As an example, we have computed this quantity for the lines $H\alpha$ and $H\beta$ (for $T = 20,000^\circ$). The results of the computation, in arbitrary units, are given in Fig. 1.

The figure shows that the main part of the energy in the line $H\beta$ is radiated by deeper layers than is the case with $H\alpha$. For example, half of the energy in $H\beta$ is radiated within a sphere of radius $r = 3r_0$, but half of the energy in $H\alpha$ is radiated within a sphere of

radius $r = 6r_0$. This conclusion is important in explaining the observed stratification of hydrogen radiation in extended atmospheres (see, for example, the paper of Goedicke on the star W Cephei³ and the paper of Baldwin on the star γ Cassiopeiae⁴).

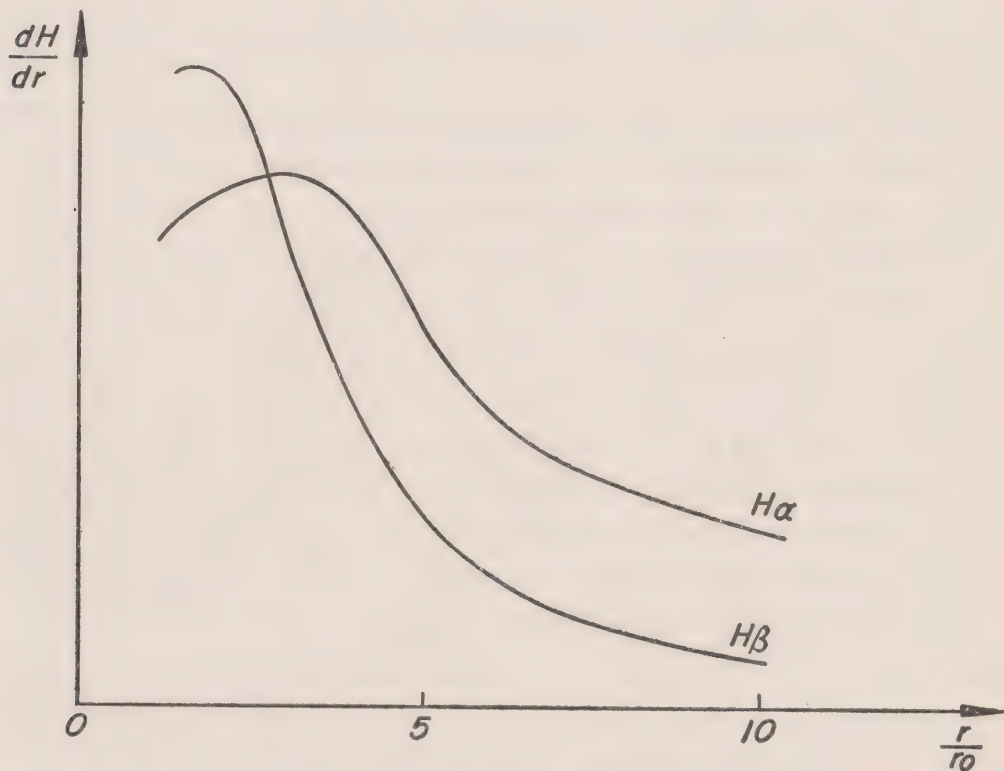


Fig. 1. Amount of energy in the Balmer line of order k , depending on the distance from the center.

In order to make a similar investigation for a given star it is necessary to know the distribution of densities and velocities in the atmosphere of the star. The important problem of how to get these data from observation can be solved by comparing the observed and theoretical contours of spectral lines. We dedicate the following section to the theoretical determination of the contours of lines which are formed within moving atmospheres.

3. Contours of Spectral Lines

The question of contours of spectral lines which are formed by the moving atmospheres of stars has been considered by many authors. In all this work it was assumed that the dark components

of bright lines occur in a reversing layer that lies beneath the transparent envelope in which bright lines are formed. In the preceding chapter it has, however, been shown that such a point of view is incorrect. In reality the dark lines and the bright lines are produced in the same extended reversing layer. Therefore, it is necessary to give a theory of contours of spectral lines that is based on this new point of view.

Generally speaking, the question on finding contours of lines within moving envelopes is much more complicated than it is for stationary ones. In one limiting case, however, when the velocity of motion of the envelope much exceeds the mean thermal velocity of atoms, contours of lines can be computed with ease. This happens because in such a case we can neglect all the factors which influence the contour of line except one—the motion of the envelope. Also, only in this case can the inverse problem—to deduce the parameters which determine the motion of the envelopes from the contours of lines—be solved with sufficient accuracy. Probably this case is realized to a close approximation in the envelopes of WR, P Cygni, Be, and new stars (novae). We will consider them now.

For the sake of simplicity, we assume that there is a sharp boundary between the photosphere and the atmosphere of the stars. In contrast to the conventional terminology, we understand by an absorption line a line which is formed by the absorption of light which is coming from the photosphere, and by an emission line, a line which is formed by the absorption of light in the atmosphere itself (taking self-reversal into account). The superimposition of the second line on the first gives the observed spectral line.

Obviously, the computation of contours of absorption lines presents no difficulty. For each frequency ν we must find the corresponding surface of equal radial velocities. A part of this surface, which is in the reversing layer, eclipses some part of the disc of the star. The remaining uneclipsed part of the star gives the residual intensity of the absorption line in the frequency ν .

In order to compute the contours of emission lines we have to find for each frequency the brightness of the corresponding surface of equal radial velocities (for unlike our case of a transparent envelope, roughly speaking only the surface radiates). Hence we

have to determine the intensity of radiation which is coming from each point in the atmosphere (owing to the Doppler effect). Denoting by $J_{ik}(r)$ the intensity of radiation which corresponds to the transition $k \rightarrow i$, which originates in the atmosphere at a distance r from the center of the star, we obviously have

$$J_{ik} = \int_0^\infty \frac{n_k A_{ki} h \nu_{ik}}{4\pi \Delta \nu_{ik}} e^{-\alpha_{ik}s} ds, \quad (26)$$

or, after performing the integration,

$$J_{ik}(r) = \frac{2h\nu_{ik}^3}{c^2} \frac{1}{(g_k/g_i)(n_i/n_k) - 1}. \quad (27)$$

The quantities n_i/n_k should be given by the theory of radiative equilibrium of a moving medium as a function of r . We can use the theory that we developed earlier.

Taking into consideration the circumstances mentioned above, we shall compute in this section contours of spectral lines separately for two cases: (a) WR and P Cygni stars and novae (that is, for the cases of ejection of the matter in stars), and (b) stars of type Be (that is, for the case of ejection of matter by rapidly rotating stars).

(a) *WR, P Cygni Stars, and Novae.* As is known, the spectra of novae before maximum show absorption lines which are displaced toward the violet and without visible traces of emission. After maximum there appear bright lines, which are edged on the violet by absorption components. Lines in the spectra of WR and P Cygni stars and of the envelopes ejected by the outbursts of new stars have similar appearances. To explain all these peculiarities of moving matter ejected from the stars, it is necessary to make a detailed analysis of the contours of spectral lines.

In theoretical works of other writers, contours have been computed separately for the dark and bright lines. In order to obtain the contours of absorption lines which are formed by a moving atmosphere, the contour has been usually assumed to be the same as that formed in a stationary atmosphere, and deformations of these contours, due to the fact that different parts of the atmosphere have different velocities relative to the observer, have been determined. Such were the papers of Carroll,⁵ Gerasimovich and

Melnikov,⁶ Wilson,⁷ Vorontsov-Velyaminov,⁸ and others, which had the purpose of explaining the contours of absorption lines in the spectra of new stars before maximum. All these papers have, however, the defect that in them everything was reduced to the "observed" Doppler effect, while in reality it is necessary to introduce the Doppler effect into the equation of radiative transfer. The first such attempt (to speak more exactly, it was an attempt at the solution of the Schuster problem, taking into account the Doppler effect) was undertaken by McCrea and Mitra.⁹ Quite recently Chandrasekhar¹⁰ was also concerned with this problem. In these works account has not, however, been taken of the change of frequency of the light quantum arising from the elementary act of scattering due to the Doppler effect, which is caused by the thermal motion of the atoms. The contours of lines computed in the above-mentioned papers had little similarity to the observed ones. This result is explained not only by inaccurate allowance for the motion of the envelope, but also by the fact that the great extent of the reversing layer, which is characteristic of the above-mentioned stars, was not taken into consideration.

In computing contours of emission lines in the works of other writers, it has been assumed that the atmosphere of the star is completely transparent to the line radiation. Beals¹¹ was the first to occupy himself with these questions, and he showed that emission lines formed by the envelope consisting of atoms which are moving with constant velocity in the radial direction should have rectangular contours. Contours of emission lines in the spectra of WR and new stars are, however, as a rule rounded. Consequently it was thought that the atoms forming the envelope are not moving with constant velocity, but are accelerated or decelerated. Contours of emission lines in such cases were determined by Gerasimovich,¹² Chandrasekhar,¹³ and Wilson.⁷ It is quite obvious that by special selection of velocity gradient one can get any symmetrical contours.

As a matter of fact, and as has been said already, we have to assume that the dark lines as well as the bright lines occur in the same extended reversing layer. In order to find the contours of lines occurring in such a layer, we shall, for the sake of simplicity, assume that matter is ejected from the star with velocity v , which does not

change with distance. We find first the contours of absorption lines (in our sense).

Let r_0 and r_1 be respectively the lower and upper limits of the reversing layer. Atoms which are moving in the atmosphere at an angle θ with the observer will absorb radiation coming from the photosphere, whose frequency is

$$\nu = \nu_0 + \nu_0 \frac{v}{c} \cos \theta. \quad (28)$$

These atoms obscure an annular portion of the disk of the star, which is included between the angles θ and θ_1 , where θ_1 is determined by the condition

$$\begin{aligned} \sin \theta_1 &= \frac{r_1}{r_0} \sin \theta, & \text{if } r_1 \sin \theta < r_0, \\ \theta_1 &= \frac{\pi}{2}, & \text{if } r_1 \sin \theta > r_0. \end{aligned} \quad (29)$$

Consequently the ratio of the intensity of frequency ν in the spectral line to the intensity in the neighboring frequency in the continuum will be

$$i_\nu = 1 - \frac{\int_\theta^{\theta_1} \psi(\theta) \cos \theta \sin \theta d\theta}{\int_0^{\pi/2} \psi(\theta) \cos \theta \sin \theta d\theta}, \quad (30)$$

where $\psi(\theta)$ is the limb-darkening law of the disk from the center to the edge. With this formula the absorption-line contours are determined.

We cannot take the function $\psi(\theta)$ in its usual form, because the stars under consideration possess extended photospheres, which cause a larger darkening of the disk from the center to the limb than in the usual star. Therefore we take the function $\psi(\theta)$ in the form

$$\psi(\theta) = a + b \cos^n \theta, \quad (31)$$

where a , b , and n are certain parameters.

It is easy to see that the computation of contours of absorption lines for different values of r_1/r_0 is reduced to the computation of

contours for $r_1 = \infty$, that is, for $\theta_1 = \pi/2$. In the last case, inserting Eq. (31) in Eq. (30), and after integration, we find

$$i_\nu = \frac{(n+2)a(1-y^2) + 2b(1-y^{n+2})}{(n+2)a + 2b}, \quad (32)$$

where instead of frequency ν we introduce a quantity y which is connected with the frequency by the relation

$$\frac{\bar{\nu} - \nu_0}{\nu_0} = \frac{v}{c}, \quad \nu = \nu_0 + y(\bar{\nu} - \nu_0). \quad (33)$$

The contours of absorption lines which were computed with the above formulae are given in Fig. 2 for two laws of darkening:

$$(A) \ b = 0; \quad (B) \ \frac{b}{a} = 4, \ n = 4. \quad (34)$$

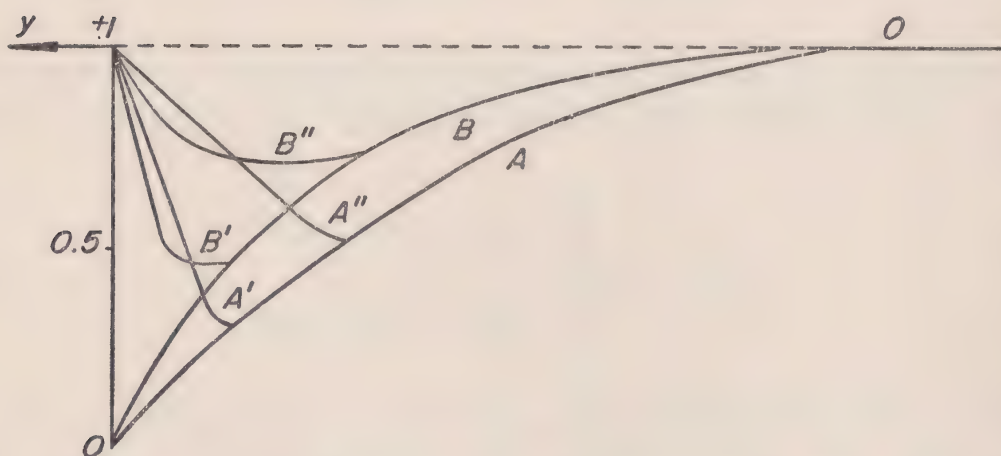


Fig. 2. Contours of absorption lines computed from two laws of limb darkening, with $n = 4$: (A) $b = 0$; (B) $b/a = 4$.

The contours for $r_1 = \infty$ are denoted by A and B . The letters A' , B' , A'' , and B'' denote contours which are computed for $r_1/r_0 = 2$, $r_1/r_0 = \sqrt{2}$, respectively.

We turn now to the determination of the contours of bright lines. In order to find the energy in frequency ν radiated by the atmosphere, we have to integrate the intensity of radiation coming out of the atmosphere over the corresponding surface of equal radial velocities. In the present case, this surface is a cone whose apex is

in the center of the star and whose axis is directed toward the observer. Therefore we obtain an expression for the energy required:

$$E_\nu = (1 - y^2) \cdot 2\pi \int_{r_0}^{r_1} J_{ik}(r) r dr, \quad (35)$$

where the intensity $J_{ik}(r)$ is determined by Eq. (27). Thus if matter is ejected with constant velocity, contours of bright lines appear to be parabolic (denoted by letter *C* in Fig. 3).

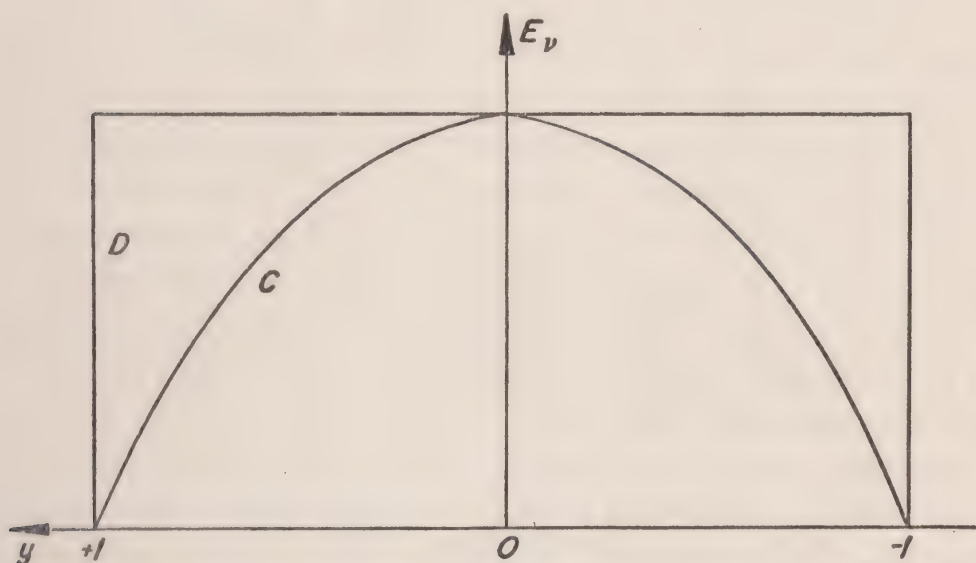


Fig. 3. Parabolic appearance of the contours of bright lines when the matter is ejected with constant velocity.

In order to obtain the contours of the spectral lines, we have to superimpose the bright line on the absorption line (examples of which we have given in Fig. 2). It thus appears that we have three possible cases, depending chiefly on the value of the integral in formula (35):

(1) Generally speaking, there appear bright lines with dark components on the violet side, i.e., lines of the appearance found in P Cygni, WR, and new stars. We have to note that the parabolic contours of bright lines appear to be much closer to the observed ones than the rectangular contour *D* (Fig. 3) obtained by Beals for the transparent envelope.

(2) If the absorption line is weak and the emission line is, on the

contrary, strong, then the second can fill in the first and we have simply a bright line. Such a case is very often realized in the spectra of WR stars (here the influence of the radiation of the transparent part of the envelope, which gives the contour D , also plays a certain part). That is why the emission lines in the spectra of these stars appear to be asymmetric and displaced toward the red. For the explanation of such red displacements Wilson¹⁵ proposed the presence of gravitational effects. We see, however, that there is no need for such a hypothesis.

(3) If the absorption line is strong and the emission is on the contrary weak, we see only the absorption line (in the usual sense). The residual intensity of the line is produced here not by the light which is scattered in the atmosphere, but by the radiation of the photosphere which is not screened by the moving atoms. Such a case has its place in the spectra of new stars before maximum brightness. As is known, those lines are sharp and narrow (in that the width of the line is considerably smaller than its displacement). Such lines are given in Fig. 2. One sees from this figure that the line edge is sharper when the thickness of the reversing layer is greater, and narrower, when the darkening of the disk from center to limb is larger. We may note that in early work devoted to the theoretical determination of contours of absorption lines in the spectra of new stars, it was assumed that the thickness of the reversing layer is small and that the law of darkening toward the limb has its usual form. The contours computed for such assumptions came out broad and shallow. Attention was first called to the importance of greater limb darkening by Vorontsov-Velyaminov,⁸ and to the importance of the large extension of the reversing layer by Mustel.¹⁴ In reality, as we have seen, both of these effects are important.

(b) *Be Stars*. As a rule, the bright lines in these stars are superimposed on the broad and shallow absorption bands. In turn, the narrow absorption lines are superimposed on the bright lines, dividing them into two components. In certain spectra, however, the bright lines appear single; in other spectra, they are both single and double.

Apparently all Be stars are stars with variable spectra. In some cases there occur variations of the total intensity of bright lines, but

sometimes the lines appear and disappear (Pleione, μ Centauri, and others); in other stars there is a periodic change of the ratio of intensities of the two components of each emission line (sometimes the red is brighter, sometimes the violet). For stars of the second group (they are called Persei type), some regularities were noted. The displacements of lines are connected with the change of intensities of the components. When the red component is stronger, the emission line and its central absorption are displaced toward the violet end of the spectrum; when the violet component is stronger, these lines are displaced toward the red. When the emission components are not equal, the broad absorption line comes out more strongly on the side of the more powerful component.

For the explanation of the contours of lines in the spectra of Be stars, O. Struve¹⁵ proposed the hypothesis that absorption bands are formed in the reversing layer of a rapidly rotating star, and that double bright lines occur in the transparent gaseous ring which is rotating about the star. Such a hypothesis is, however, completely unsatisfactory. It is necessary also to assume a process of ejection of matter, which has a variable character. On the other hand, in the opinion of McLaughlin,¹⁶ the above-mentioned changes in the spectra of Persei type can be explained by assuming a pulsation of the rotating star.

Thus the envelopes of Be stars apparently participate in two types of motion—radial and rotational. The problem therefore arises of finding the contours of spectral lines formed by such an envelope. Struve considers such a problem only for a transparent rotating ring. It is interesting, however, to solve the problem for the general case (rotation plus radial motion for a nontransparent envelope). We shall compute now the contours of lines which are formed by a rotating and expanding envelope for the limiting case mentioned above ($v \gg u$). Here we again assume that the velocity of expansion is similar in all layers. As regards the velocity of rotation, we shall assume that it decreases in inverse proportion to the distance from the center of the star, that is, we assume that angular momentum is conserved.

For the sake of simplicity we restrict ourselves to the equatorial plane of the star. Let $v'(r) = v'(r_0)r_0/r$ be the velocity of

rotation and v'' the velocity of expansion. Then the radial velocity will be

$$v = v'(r_0) \left(\frac{r_0}{r} \sin \theta + k \cos \theta \right), \quad (36)$$

where $k = v''/v'(r_0)$.

Contours of absorption lines, computed by us for two cases, $k = 0$ (pure rotation) and $k = \frac{1}{3}$, for the two laws of darkening (34) are given in Fig. 4; here the quantity y is again determined by

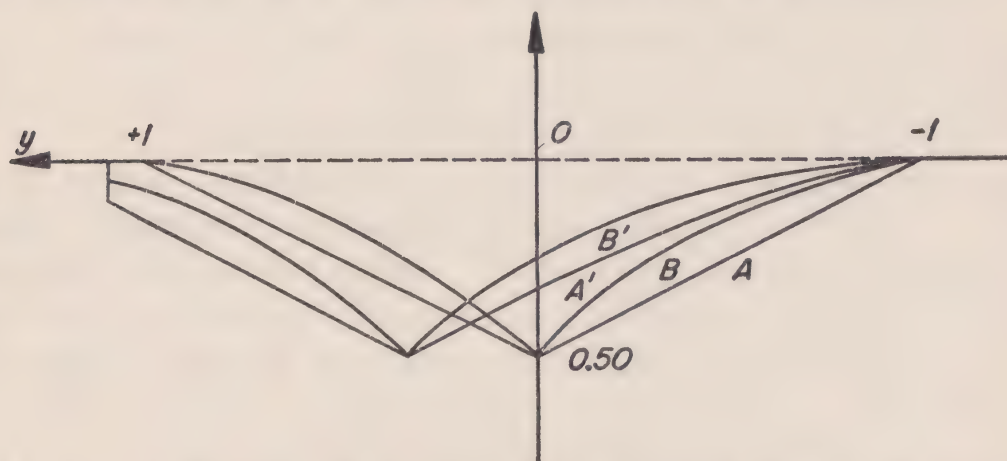


Fig. 4. Contours of absorption lines computed for $k = 0$ (pure rotation) and $k = \frac{1}{3}$ for the two laws of limb darkening given in Eqs. (34).

relation (33), in which by v we now understand the quantity $v'(r_0)$.

We find now the contours of bright lines. Let E_ν be the amount of energy in frequency ν radiated by the equatorial layer in unit solid angle. For this quantity we find

$$E_\nu = \int_{a_0}^{a_1} J_{ik}(r) da, \quad (37)$$

where $a = r \sin \theta$, and the limits of integration are determined by the relations

$$a_0 = r_0 \sin \theta_0, \quad \sin \theta_0 + k \cos \theta_0 = y, \quad (38)$$

$$a_1 = r_1 \sin \theta, \quad \frac{r_0}{r_1} \sin \theta_1 + k \cos \theta_1 = y, \quad (39)$$

$$a_1 = \frac{2}{k} r_0 \left\{ \frac{y}{k} \pm \left[\left(\frac{y}{k} \right)^2 - 1 \right]^{\frac{1}{2}} \right\}, \quad (40)$$

where we take one of the relations (39) or (40), depending on whether the curves of equal radial velocity cross the circumference of radius r_1 or not. The quantity $J_{ik}(r)$ is again determined by formula (27).

In order to compute the integral (37), we have to know the dependence of the quantities n_i/n_k on the radius, for which in the present case it is necessary to express the parameter x , on which the relation n_i/n_k depends, as a function of r (see the preceding section). For simplicity we assume $J_{ik} = \text{const}$. Then instead of Eq. (37) we have

$$E_v = J_{ik}(a_1 - a_0). \quad (41)$$

In the case of pure rotation this reduces to the form

$$\begin{aligned} E_v &= J_{ik}r_0y \left[\left(\frac{r_1}{r_0} \right)^2 - 1 \right], \quad \text{if } |y| < \frac{r_0}{r_1}; \\ E_v &= J_{ik}r_0y \left[\frac{1}{y^2} - 1 \right], \quad \text{if } |y| > \frac{r_0}{r_1}. \end{aligned} \quad (42)$$

The contours of bright lines computed in accordance with Eqs. (41) and (42) for the cases $k = 0$ ($r_1/r_0 = 2$, $r_1/r_0 = 4$) and $k = \frac{1}{3}$ ($r_1/r_0 = 2$), are given in Fig. 5. These lines should be superimposed on the absorption lines whose examples we gave in Fig. 4.

Looking at the above formulae and graphs, we can make the following conclusions:

(1) Generally speaking, the contours are very similar to those of the lines in the Be stars. In the case of pure rotation, both components of the bright lines appear to be similar, and in the case of rotation and expansion, the bright lines are displaced to the violet, and the violet component is weaker than the red one. This very kind of line is often seen in the spectra of Be stars. This fact witnesses the ejection of matter in these stars also.

(2) The computed bright lines usually appear double. The distance between the two components is, however, smaller, the smaller is the rotation of the star, and the larger is the quantity r_1/r_0 . Consequently the components in some stars should coalesce and we should see single lines. On the other hand, in the spectra of other

stars we can see single as well as split lines (since for these lines the quantity r_1/r_0 is different). Thus the differences in bright lines in the spectra of Be stars are apparently explained.

(3) In the case of an extended reversing layer, the total intensity of an absorption line increases with the velocity of rotation. This fact should be noted especially because in the conventional case the

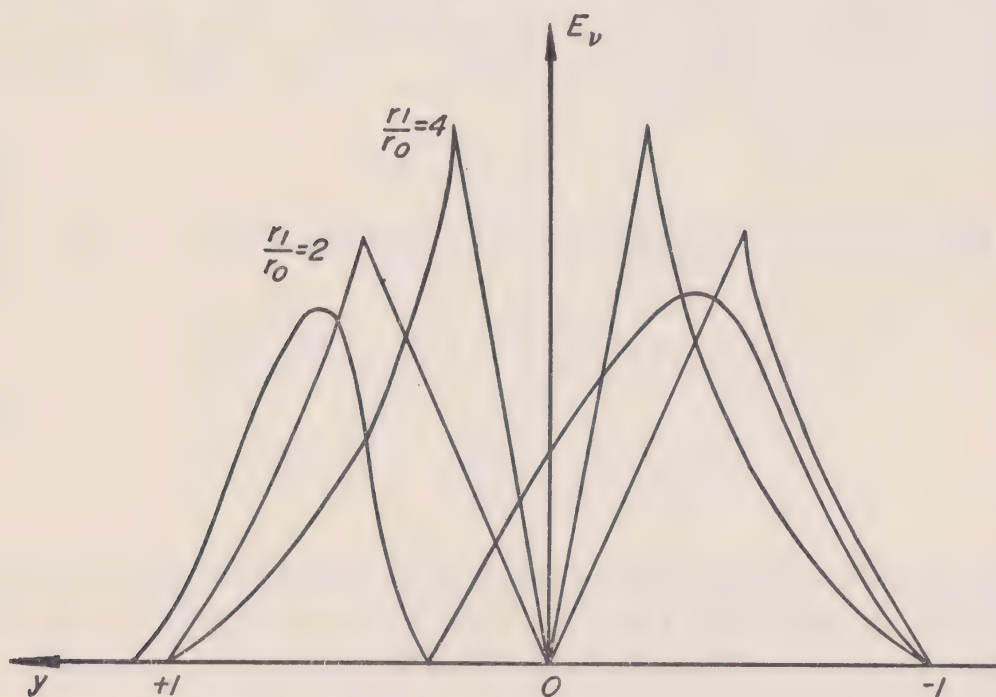


Fig. 5. Contours of emission lines for $k = 0$ ($r_1/r_0 = 2, 4$) and $k = \frac{1}{3}$ ($r_1 r_0 = 2$).

total intensity of an absorption line does not change with widening of the line due to the rotation of the star.

We see that the proposed theory of contours of lines in the spectra of WR and P Cygni stars and novae is found to be in good agreement with observation, despite its elementary character. This speaks in favor of the assumed concept of the motion of stellar atmosphere, as well as in favor of our point of view about the origin of bright as well as dark lines in one and the same extended envelope. It seems to us that in the future there will be great interest in applying the results obtained in this chapter to the study of actual stars, for example, P Cygni, γ Cassiopeiae, and so forth. Owing to the simplicity of the results obtained, this will not present any difficulties.

III

GASEOUS NEBULAE

Now we consider the envelopes whose optical depth, beyond the limit of the fundamental series, considerably exceeds unity. In such envelopes, the chief part in the ionization of atoms from the ground state is played by radiation coming not directly from the star, but from the diffused radiation of the envelope itself. The appearance of this new unknown quantity, the density of this diffused radiation, leads to a serious difficulty.

For the gaseous nebulosities which we consider in this chapter, these difficulties appear to be minimal. These objects are characterized by two properties: (1) a relatively small number of ionizations from excited states, and (2) complete transparency to radiation in lines of subordinate series. Consequently, the field of investigation of their radiation can be divided into two steps. First we investigate the radiation field beyond the limit of the fundamental series (thus we find the degree of ionization), and then we study the radiation field of resonance lines (thus we find the relative number of atoms in the second state). Concerning the radiation in lines of subordinate series, the intensity of this radiation is determined, as was said above, by means of the equations of Cillié.

As is known, the radiation field of nebulosity was first considered by Ambartsumian¹ in his important work on the radiative equilibrium of planetary nebulae. Later this question was discussed in the works of Chandrasekhar,² Hagihara,³ and Menzel and his collaborators.⁴ In all these papers it was assumed, however, that the nebula is stationary or that it moves without velocity gradient (the second case does not differ in principle from the first, the difference between them consisting only in writing down the boundary conditions). Zanstra⁵ was the first to make an attempt to find the density of $L\alpha$ radiation in a nebula which is moving with a velocity gradient. This attempt was, however, unsuccessful, as will be shown later.

The contents of this chapter are as follows. In Sec. 1 we consider the field of L_c radiation (for clarity we discuss a hydrogen nebula). In Sec. 2 we shall find the density of $L\alpha$ radiation and light pressure caused by this radiation for nebulae moving with velocity gradient. In Sec. 3 we discuss the question of the temperatures of nebulae and exciting stars.

1. The Field of L_c Radiation

In accordance with the work of Milne,⁶ the following model of a planetary nebula (and of any other nebular envelope surrounding a star) is universally accepted. A planetary nebula is a spherical envelope, the thickness of which is very small in comparison with its distance from the center of the star. Consequently, the envelope may be assumed to consist of plane parallel layers, and the coefficient of dilution in the envelope is constant.

Let κ_{1c} be the average absorption coefficient of an atom beyond the Lyman limit, and let τ be the corresponding optical depth, reckoned from the internal boundary of the envelope. Let us further call πS_{1c} the total number of L_c quanta which fall from the star on 1 cm^2 of the internal boundary of the envelope. Let $K_{1c}(\tau, \theta)$ be the number of quanta of diffused L_c radiation which are going through at depth τ and with angle θ toward the external normal in the unit solid angle through the unit surface which is perpendicular to the rays.

Because the number of ionizations coming from the ground state should be equal to captures in all states, we obtain

$$n_1 \kappa_{1c} \int K_{1c} d\omega + n_1 \kappa_{1c} \pi S_{1c} e^{-\tau} = n_e n^+ \sum_1^{\infty} C_i. \quad (1)$$

Denoting by p the fraction of captures in the first state, and introducing the quantity C_{1c} which is determined by

$$n_e n^+ C_1 = 4\pi n \kappa_{1c} C_{1c}, \quad (2)$$

we find instead of Eq. (1),

$$C_{1c} = p \int K_{1c} \frac{d\omega}{4\pi} + p \frac{S_{1c}}{4} e^{-\tau}. \quad (3)$$

On the other hand, the quantities C_{1c} and K_{1c} are connected with one another by the usual equation of radiative transfer:

$$\cos \theta \frac{dK_{1c}}{d\tau} = C_{1c} - K_{1c}. \quad (4)$$

In addition, the following boundary conditions always hold:

$$\begin{aligned} K_{1c}(0, \theta) &= K_{1c}(0, \pi - \theta), \\ K_{1c}(\tau_1, \theta) &= 0 \quad \left(\text{for } \theta > \frac{\pi}{2} \right) \end{aligned} \quad (5)$$

(the condition for the internal boundary takes into account the diffused radiation which comes from the opposite side of the nebula).

Thus the problem of determining the degree of ionization in a nebula reduces to the solutions of Eqs. (3) and (4) with the boundary condition (5).

Out of these equations we obtain the following integral equation for the determination of the quantity $C_{1c}(\tau)$:

$$C_{1c}(\tau) = \frac{p}{2} \int_0^{\tau_1} [E_i |\tau - \tau'| + E_i(\tau + \tau')] C_{1c}(\tau') d\tau' + p \frac{S_{1c}}{4} e^{-\tau}. \quad (6)$$

For $\tau = \infty$ and at large optical depth, instead of Eq. (6) we obtain

$$C_{1c}(\tau) = \frac{p}{2} \int_{-\infty}^{+\infty} E_i |\tau - \tau'| C_{1c}(\tau') d\tau'. \quad (7)$$

This equation has an exact solution:

$$C_{1c}(\tau) = A e^{-k\tau}, \quad (8)$$

where k is the root of:

$$\frac{p}{2k} \ln \frac{1+k}{1-k} = 1 \quad (9)$$

and A is an arbitrary constant. We shall consider Eq. (8) as an approximate solution of Eq. (6), and shall find the constant A from the condition that Eq. (6) should be correct on the average. Then we obtain A from

$$A = \frac{kpS_{1c}}{4(1-p)}. \quad (10)$$

The quantity p appears to be a function of the electron temperature. For a planetary nebula we can adopt $p = \frac{1}{2}$. Then Eq. (9) gives $k = 0.96$. We see that the expression found for $C_{1c}(\tau)$ differs relatively little from the expression which we obtain by taking into account only direct radiation coming from the star. Consequently we may say that the degree of ionization in the nebula varies according to the law

$$n_e \frac{n^+}{n_1} = W \left(\frac{T_e}{T_*} \right)^{\frac{1}{2}} \frac{(2\pi mkT_*)^{\frac{3}{2}}}{h^3} e^{-h\nu_{1c}/kT_*} 2e^{-\tau}. \quad (11)$$

We should note two points:

(1) By the approximate solution of Eqs. (3) and (4) we obtain $k = 2(1 - p)^{\frac{1}{2}}$ (method of Schwarzschild and Schuster) and $k = [3(1 - p)]^{\frac{1}{2}}$ (method of Eddington). The difference between these values of k and that which we obtain from Eq. (9) is undoubtedly important in many problems. For the planetary nebulae, however, this approximate method does not lead to any serious errors because the chief role is played by the direct radiation.

(2) Above, we represented all the Lyman continuum by one equation in which we adopted a certain mean absorption coefficient. A more accurate treatment of the question was given by Menzel and his collaborators,⁴ who derived an equation of radiative equilibrium for each interval of frequency (taking into account not only photoionizations and captures, but also electron collisions). However, it is easy to see that there is no need for this procedure because the energy radiated by an elementary volume of the nebula beyond the Lyman limit is a known function of frequency: $\varepsilon_\nu \sim e^{-h\nu/kT_e}$. Therefore by a more accurate consideration of the problem we come again to an integral equation of type (6) (with a little more complicated integrand) with one unknown function depending only on τ .

2. The Field of L_α Radiation

When considering the field of L_c radiation, it is not important to us whether the envelope is in motion or not (so long as the velocity of the envelope is small in comparison with the velocity of light). We now allow the envelope to move (let us say, to expand) with a

velocity gradient. For simplicity, let us assume that $dv/d\tau$ is constant and that $dv/d\tau > 0$.

As before, we now assume that the absorption coefficient in the lines is constant in the interval $\Delta\nu_{12}$ and equal to zero outside this interval. Owing to the presence of the velocity gradient, some fraction of the L_α quanta which enter a given part of the envelope (whose frequency does not fall within the interval indicated) will not be absorbed at that place. Consequently the integral equation which determines the density of the L_α radiation will have a completely different form from what it has in the absence of a velocity gradient.

In endeavoring to form this equation we agree first that, when speaking of the density of L_α radiation, we understand only those L_α quanta found in the given volume that can be absorbed in that volume.

We begin with the condition of stationaryness for the second state of the hydrogen atom:

$$n_2 A_{21} = n_1 B_{12} \rho_{12} + \sum_3^{\infty} n_k A_{k2} + n_e n^+ C_2. \quad (12)$$

Because the envelope is transparent to line radiation in subordinate series, we have

$$\sum_3^{\infty} n_k A_{k2} = n_e n^+ \sum_3^{\infty} C_k. \quad (13)$$

Inserting Eq. (13) in Eq. (12) and taking into account the results of the foregoing section, we obtain

$$n_2 A_{21} = n_1 B_{12} \rho_{12} + n_1 \kappa_{1c} k \pi S_{1c} e^{-k\tau}. \quad (14)$$

Assuming that

$$\frac{k}{4} S_{1c} e^{-k\tau} = \psi(\tau) \quad (15)$$

and

$$n_2 A_{21} = 4\pi n_1 \kappa_{12} C_{12}, \quad n_1 B_{12} \rho_{12} = n_1 \kappa_{12} \int K_{12} d\omega, \quad (16)$$

where K_{12} is the mean absorption coefficient of the atom in the line L_α , instead of Eq. (13) we have

$$C_{12} = \int K_{12} \frac{d\omega}{4\pi} + q\psi(\tau), \quad (17)$$

where $q = \kappa_{10}/\kappa_{12}$.

In the case under consideration, that is, in the presence of a velocity gradient in the envelope, the second equation which connects the quantities C_{12} and K_{12} can no longer be taken from the usual equation of radiative transfer. Instead of this equation, and by means of the very same considerations which were presented in Chapter I, we obtain the following more complicated integral relation:

$$K_{12}(t, \theta) = \int_0^t C_{12}(t') e^{-(t-t') \sec \theta} [1 - \beta(t-t') \cos \theta] \sec \theta dt' \quad (\theta < \pi/2), \quad (18)$$

where t is the optical depth in the line L_α , and β is, as before,

$$\beta = \frac{g}{2u} \frac{dv}{d\tau}. \quad (19)$$

We have neglected here the radiation in the line L_α which falls on the internal boundary of the envelope, from the star as well as from the opposite parts of the envelope (this means that the velocity of expansion of the internal boundary is assumed to be large enough in comparison with the mean thermal velocity of the atoms).

Inserting Eq. (18) (and the analogous relation for $\theta > \pi/2$) in Eq. (17), we obtain the following integral equation which determines the quantity C_{12} :

$$C_{12}(t) = \frac{1}{2} \int_0^{t_1} C_{12}(t') [E_i |t - t'| - \beta E_{i2} |t' - t| \cdot |t' - t|] dt' + q\psi(\tau), \quad (20)$$

where

$$E_{ix} = \int_1^\infty e^{-xz} \frac{dz}{z}, \quad E_{izx} = \int_1^\infty e^{-xz} \frac{dz}{z^2}. \quad (21)$$

Because we assume that the optical depth τ_1 of the nebula beyond the limit of the Lyman series is larger than unity, the optical depth t_1 in L_α must be larger than 10^4 . Therefore for the mean path in the nebula the limits of integration of Eq. (20) can be replaced by infinity. It is easy to be convinced that the solution thus obtained has the form

$$C_{12}(t) = Ae^{-kt} + Be^{kt} + 3 \frac{q}{\beta} \psi(\tau), \quad (22)$$

where k is the root of the equation

$$\frac{1}{2k} \ln \frac{1+k}{1-k} - \frac{\beta}{k^2} \left[1 + \frac{1}{1+k^2} - \frac{1}{k} \ln \frac{1+k}{1-k} \right] = 1 \quad (23)$$

and A and B are arbitrary constants.

In Eq. (22) we shall find the solution of Eq. (20) by determining the constants A and B from the boundary conditions. Inserting Eq. (22) in Eq. (20) and assuming that t is small, we obtain

$$Ae^{kt} \int_t^\infty Eixe^{kx} dx + \frac{3}{\beta} q\psi(0) \int_t^\infty Eidx = 0. \quad (24)$$

This condition being satisfied on the average, we find

$$A = \frac{3}{2} \frac{q}{\beta} \frac{k^2}{\ln(1-k) + k} \psi(0). \quad (25)$$

Analogously we obtain for B :

$$B = \frac{3}{2} \frac{q}{\beta} \frac{k^2}{\ln(1-k) + k} \psi(\tau_1) e^{-kt_1}. \quad (26)$$

Because β is small, it follows approximately from Eq. (23) that

$$k = \sqrt{\beta}. \quad (27)$$

Therefore the final solution of Eq. (20) can be written in the form

$$C_{12}(t) = 3 \frac{q}{\beta} \left\{ \psi(\tau) - \left(1 - \frac{2}{3} \sqrt{\beta} \right) [\psi(0) e^{-t\sqrt{\beta}} + \psi(\tau_1) e^{-(t_1-t)\sqrt{\beta}}] \right\}. \quad (28)$$

In addition to the quantity $C_{12}(t)$, we are interested in the flux of L_α quanta, which determines the radiation pressure in the nebula. Neglecting the term with the factor β , we obtain for the flux

$$F(t) = 2\pi \left[\int_0^t E_{i2}(t - t') C_{12}(t') dt' - \int_t^{t_1} E_{i2}(t' - t) C_{12}(t') dt' \right]. \quad (29)$$

For the interior of the nebula this formula gives

$$F(t) = -4\pi \frac{q^2}{\beta} \frac{d\psi}{d\tau}. \quad (30)$$

At the boundary, the flux is given by

$$F(0) = -2\pi\psi(0) \frac{q}{\sqrt{\beta}}, \quad (31)$$

$$F(t_1) = +2\pi\psi(\tau_1) \frac{q}{\sqrt{\beta}}.$$

We now consider for comparison a nebula which is expanding without velocity gradient. In this case, in Eq. (20) one must put $\beta = 0$. Then for the mean path in the nebula, the solution of Eq. (20) will have the form

$$C_{12}(t) = 3\psi(0) \frac{t_1 - t}{qt_1}. \quad (32)$$

For the flux of L_α radiation at the internal boundary of the nebula, we obtain without computation

$$F(0) = -4\pi\psi(0) \quad (33)$$

(because in the absence of velocity gradient the flux of L_α quanta at the internal boundary must be equal to the flux of L_c quanta coming from the central star).

Comparing Eqs. (32) and (33) with Eqs. (28) and (31) respectively, we see that the appearance of a velocity gradient in the nebula diminishes the density and the flux of L_α radiation very sharply. Assume, for example, that the velocities $dv/d\tau$ and u are of the same order. Then the first of Eqs. (31) gives for the flux of L_α radiation at the internal boundary of the nebula a value many hundred times smaller than that given by Eq. (33), and for the radiation density of

L_α in the mean parts of the nebula we obtain by Eq. (28) a value which is several thousand times less than that derived from Eq. (32).

As has already been pointed out, the field of L_α radiation in a nebula moving without velocity gradient was first considered by Ambartsumian. In this work the great importance of radiation pressure due to L_α radiation was clearly demonstrated. Computations have shown that at the internal boundary of the nebula the light pressure is approximately 1000 times greater than the gravitation of the central star. This result is important on two accounts: first, because it opens new perspectives in making a dynamical study of the nebulae, taking into account only one force—light pressure due to L_α radiation; second, because Ambartsumian pointed out the strong retardation of the internal parts of the nebula, and thus strengthened the hypothesis of the origin of planetary nebulae from the envelopes of supernovae. (As is known, the envelopes of supernovae are ejected with great velocities, of the order of several thousand kilometers per second, and planetary nebulae expand with a velocity of the order of several tens of kilometers per second; the explanation of this difference by gravitational retardation alone appears to be untenable.)

In reality, however, we must think that nebulae move with a velocity gradient (for, even if at some moment the nebula moved with constant velocity in all layers, in time, owing to radiation pressure, there would be differences of expansion). But the appearance of a velocity gradient, as has been shown above, sharply diminishes the flux of L_α radiation. Therefore we come to the conclusion that light pressure due to L_α radiation does not play as great a part as was previously ascribed to it.

We should note, however, that the problem solved in this section has been previously considered by Zanstra.^{5 6} The method of Zanstra is, however, different from ours. Zanstra, by considering a nebula bounded by plane parallel layers, took into account only that component of the velocities of the atoms that was perpendicular to these boundaries. He assumed approximately that, if this component is equal for all atoms, then they can absorb radiation from each other, and if it differs, they cannot.

It is clear that with such assumptions the fast-moving atoms are

able to absorb (and therefore radiate) only a very small amount of energy because they are unable to absorb the radiation coming from the side. Therefore the line radiation produced in an elementary volume will have not a rectangular but a parabolic contour. This difference appears to be very fundamental, because otherwise the fraction of the nonabsorbed energy will be considerably smaller than in the first case. Consequently the density and flux of L_α radiation obtained by the method of Zanstra must be very different from ours.

This is so in reality. Taking Zanstra's value for the flux of L_α radiation from the book of Ambartsumian,⁷

$$F(t) = \frac{\pi}{3} \left(\frac{q}{\beta} \right)^2 \frac{d\psi}{d\tau}, \quad (34)$$

and comparing it with ours from Eq. (30), we see that the difference is terrific. For example, if we assume that the velocities u and $dv/d\tau$ are of the same order, the Zanstra flux will be about 10,000 times larger than ours.

The reason for the discrepancy is that Zanstra did not take into account the change of frequency of the light quanta which is due to the Doppler effect in the elementary act of scattering. That is, he assumes that scattering is coherent. The error is apparent in considering a moving envelope in which the Doppler effect plays an important part. However, the noncoherence of light scattering plays an important part also for the stationary atmosphere. In view of the importance of this question, we shall make a special investigation of the problem in what follows.

3. Temperatures of Nebulae and Nuclei

In the previous sections we have considered the condition of stationariness for the first and the second stages of the hydrogen atoms. Now we shall consider one additional condition of stationariness, the law of conservation of energy for the free electrons in the nebula.

We shall assume that free electrons receive energy by photoionization of the hydrogen atoms, and lose it in three ways: (1) by

capture of protons and by free-free transitions, (2) by collisions which excite the lines N_1 and N_2 , and (3) by inelastic collisions with hydrogen atoms. Obviously if by mathematical presentation of this law we are able to limit ourselves only to the values known from observation, we shall be able to obtain a very important relation connecting the temperature of the central star and the electron temperatures of the nebulae. Let ε be the mean energy received by an electron by photoionization. Because the number of ionizations must be equal to the number of recombinations, the amount of energy which is obtained by the electrons per cubic centimeter per second will be $\varepsilon n_e n^+ \sum_1^{\infty} C_i$. Part of this energy, which is lost by recombinations and free-free transitions, we denote by

$$n_e n^+ \left(\sum_1^{\infty} C_i \varepsilon_i + f \right),$$

and another part, which is transformed into radiation in the nebular lines, by E , and finally the third part, which is expended by collisions with hydrogen atoms, by $n_1 n_e \left(\sum_2^{\infty} D_i h\nu_{1i} + D_c h\nu_{1c} \right)$.

The law of conservation of energy gives

$$\varepsilon n_e n^+ \sum_1^{\infty} C_i = n_e n^+ \left(\sum_1^{\infty} C_i \varepsilon_i + f \right) + E + n_1 n_e \left(\sum_2^{\infty} D_i h\nu_{1i} + D_c h\nu_{1c} \right). \quad (35)$$

Here E is unknown, because we know neither the effective cross sections for the collisions of electrons with O III atoms nor the concentration of these atoms. But these quantities can be found in another way, namely, if we assume that the ratio of intensities of the lines $N_2/H\beta$ is known.

Taking into account that $N_1/N_2 = 3$, we find

$$\int E dV = 4 \frac{N_2}{H\beta} A_{42} h\nu_{24} \phi_4(T_e) \int n_e n^+ dV, \quad (36)$$

where the integration is extended over the whole volume of the nebula, and $\phi_4 = n_4/n_e n^+$, and this is the quantity which is determined from Eqs. (1.12) of Cillié.

Integrating Eq. (35) over the volume of the nebula, and taking into consideration Eq. (36), we obtain

$$\bar{\varepsilon} \sum_1^{\infty} C_i = \sum_1^{\infty} C_i \varepsilon_i + f + 4 \frac{N_2}{H\beta} A_{42} h\nu_{24} \phi_4 + \frac{\bar{n}_1}{n^+} \left(\sum_2^{\infty} D_i h\nu_{1i} + D_c h\nu_{1c} \right), \quad (37)$$

where the following notations are introduced:

$$\bar{\varepsilon} = \frac{\int \varepsilon n_e n^+ dV}{\int n_e n^+ dV}, \quad (38)$$

$$\frac{\bar{n}_1}{n^+} = \frac{\int n_1 n_e dV}{\int n^+ n_e dV}. \quad (39)$$

The quantity $\bar{\varepsilon}$ is the energy of electrons obtained by photoionization, which is averaged for the whole nebula. Generally speaking, it depends on the temperature of the central star and on the optical thickness of the nebula beyond the Lyman limit. We consider two limiting cases. First, the nebula absorbs only a small part of the stellar radiation ($\tau_1 \ll 1$); second, the nebula absorbs the whole energy of the star beyond the Lyman limit ($\tau_1 \gg 1$).

In the first case we can assume that ε is determined only by the ionizing radiation which comes directly from the star. Therefore, assuming that

$$\bar{\varepsilon} = AkT_*, \quad (40)$$

where k is the Boltzmann constant, we find for the quantity A :

$$A = \frac{\int_{x_0}^{\infty} (e^x - 1)^{-1} dx}{\int_{x_0}^{\infty} [x(e^x - 1)]^{-1} dx} - x_0, \quad (41)$$

where $x_0 = \frac{h\nu_{1c}}{kT_*}$.

In the second case the photoionization is due to the radiation directly from the star and also to the diffused radiation of the nebula itself. With large τ , we can think, however, that all quanta radiated by the captures of electrons in the first level are absorbed by the nebula itself, that is, the number of ionizations under the influence of the diffused radiation is equal to $C_1 \int n_e n^+ dV$, and the energy which electrons receive by this process is equal to $C_1 \epsilon \int n_e n^+ dV$. Therefore in this case also the diffused radiation of the nebula may be neglected. We have only to sum C_i and $C_i \epsilon_i$, not from the first but from the second level. For A we find

$$A = \frac{\int_{x_0}^{\infty} x^3 (e^x - 1)^{-1} dx}{\int_{x_0}^{\infty} x^2 (e^x - 1)^{-1} dx} - x_0. \tag{42}$$

The quantity A , computed from Eqs. (41) and (42), is given in Table 11.

Table 11. Values of A computed from Eqs. (41) and (42)

$T_*/1000$	Eq. (41)		Eq. (42)	
	A	$AT_*/1000$	A	$AT_*/1000$
20	0.90	18	1.24	25
40	.83	33	1.46	58
60	.77	46	1.63	98
80	.71	57	1.76	141

From this table we see that for the given interval of stellar temperatures the energy $\bar{\epsilon}$ is approximately twice as large in the second case as in the first. Because the number of captures for the first level is about half of the total number of captures, Eq. (37) should in both cases give nearly the same results. In the future we shall consider only the second case, which is nearer to the real nebulae.

Supplementing Eq. (40), we write

$$\sum_2^{\infty} C_i \varepsilon_i + f = BT_e k \sum_2^{\infty} C_i, \quad (43)$$

$$4A_{42}h\nu_{24}\phi_4 = Ck \sum_2^{\infty} C_i, \quad (44)$$

$$\sum_2^{\infty} D_i h\nu_{1i} + D_c h\nu_{1c} = Dk \sum_2^{\infty} C_i. \quad (45)$$

Then instead of Eq. (37) we obtain

$$AT_* = BT_e + C \frac{N_2}{H\beta} + D \frac{\bar{n}_1}{n^+}. \quad (46)$$

This is the final form of the relation which connects the temperature of the nucleus with the electron temperature of the nebula. In order to apply this relation to actual nebulae it is necessary still to compute the coefficients B , C , and D and also to give formulae for finding the mean degree of ionization in the nebula.

To compute the coefficients B , C , and D we have the following formulae. According to Cillié,⁸ the energy radiated per cubic centimeter per second, for free-free transitions, is equal to

$$n_e n^+ \frac{2^7 \pi^3}{(6\pi)^{\frac{3}{2}}} \left(\frac{kT_e}{m} \right)^{\frac{1}{2}} \frac{e^6}{hc^3 m}, \quad (47)$$

and the energy radiated by the captures at the i th level is equal to

$$n_e n^+ \frac{2^9 \pi^5}{(6\pi)^{\frac{3}{2}}} \frac{e^{10}}{mc^3 h^3} \left(\frac{m}{kT_e} \right)^{\frac{1}{2}} \frac{1}{i^3}. \quad (48)$$

In our notation, the first of these formulae is $n_e n^+ f$ and the second is $n_e n^+ C_i (\varepsilon_i + h\nu_{ic})$.

According to Shōtarō Miyamoto,⁹ we find for the quantity D_i

$$D_i = \bar{Q}_i \cdot 4 \left(\frac{kT_e}{2\pi m} \right)^{\frac{1}{2}} \left(1 + \frac{h\nu_{1i}}{kT_e} \right) e^{-h\nu_{1i}/kT_e}, \quad (49)$$

where \bar{Q} is the mean effective cross section for the excitation of the i th state from the ground state. We can assume that for $i = 2, 3, 4, 5$

$\bar{Q}_i = 1.0, 0.2, 0.1, 0.06$, respectively. Here the quantity \bar{Q}_i is expressed in units of $\pi a_0^2 = 0.88 \cdot 10^{-16} \text{ cm}^2$. The quantity D_c is determined by a formula analogous to Eq. (49), where $\bar{Q} = 0.2$ and the energy of ionization $h\nu_{1e}$ appears instead of $h\nu_{1i}$.

The coefficients B , C , and D computed by the above-mentioned formulae are given in Table 12.

Table 12. Values of the coefficients B , C , and D as functions of temperature

$T_e/1000$	B	$BT_e/1000$	$C/1000$	$D/1000$
5	1.02	5	12	0.001
7.5	1.04	8	12	3.0
10	1.06	11	12	250
12.5	1.08	14	12	2.500
15	1.10	17	12	16.000

From Table 12 we see among other things that the coefficient D changes very rapidly with the electron temperature. This means that the quantity \bar{n}_1/n^+ , which appears in Eq. (46), need be known only approximately. We can, for example, indicate the following method for finding this quantity.

Assuming that the degree of ionization in the nebula varies according to the law of Eq. (11), we obtain

$$\frac{\bar{n}_1}{n^+} = \left(\frac{n_1}{n^+} \right)_0 \ln \left(\frac{n^+}{n_1} \right)_0, \tag{50}$$

where $(n^+/n_1)_0$ is the degree of ionization which is determined by the usual formula, that is, by means of formula (11) with $\tau = 0$. In order to find the quantity $(n^+/n_1)_0$ we have to know the coefficient of dilution and the density of electrons for $\tau = 0$. Using the well-known formula

$$M_* = m_* + 5 - 5 \log r, \tag{51}$$

$$M_* = -5 \log \frac{r_*}{r_0} + \frac{36700}{T_*} - 0.72, \tag{52}$$

we find the coefficient of dilution

$$\log W = -7.00 - 0.4m_* - 2 \log D'' + \frac{14700}{T_*}, \quad (53)$$

where m_* is the photographic magnitude of the nucleus and D'' is the angular diameter of the nebula in seconds of arc. The other interesting quantity—the number of free electrons per cubic centimeter for $\tau = 0$ —can be found from the formula

$$n_e = \left(\frac{E\beta}{V\phi_4 A_{42} h\nu_{24}} \right)^{\frac{1}{2}}, \quad (54)$$

where $E\beta$ is the energy radiated by the nebula in the line $H\beta$. Assuming that $H\beta$ radiated only one 1/15 of the total energy radiated by the nebula in the visual part of the spectrum and that the thickness of nebula is equal to 1/10 of its radius, we find instead of Eq. (54)

$$\log n_e = 8.88 - \frac{3}{2} \log D'' - 0.2 m_n - \frac{1}{2} \log r, \quad (55)$$

where m_n is the apparent magnitude of the nebula and r is the distance of the nebula in parsecs. Thus the average degree of ionization in the nebula can be determined from formulae (50), (11) (for $\tau = 0$), (53), and (55).

We can use the relation (46) for finding the electron temperature of the nebula provided the temperature of the nucleus is known. Methods for determination of the temperatures of nuclei were first given by Zanstra.¹⁰ One of these methods (the method of “nebulium”) is based on the assumption that the whole energy received by the free electrons due to photoionization is expended on the excitation of the lines of “nebulium.” This method yields the temperature of the nucleus from the relations

$$\int_{x_0}^{\infty} \frac{x^2(x - x_0)}{e^x - 1} dx = \sum_{\text{neb}} \frac{x_k^4}{e^{x_k} - 1} A_k, \quad (56)$$

where the quantities A_k (which were obtained from the observations) are equal to

$$A_k = \frac{E_k}{\nu E_\nu^*}, \quad (57)$$

where E_k is the total energy radiated by the nebula in a given line,

and E_p^* is the energy radiated by the nucleus in a monochromatic interval of frequency at the same place of the spectrum. In reality, only a fraction of the energy of free electrons is expended in the excitation of lines of “nebulium.” According to Eq. (46) this fraction is equal to $(C/AT_*)(N_2/H\beta)$. Therefore instead of Eq. (56) we obtain a more accurate relation for determination of temperature of the nucleus:

$$\frac{C}{AT_*} \frac{N_2}{H\beta} \int_{x_0}^{\infty} \frac{x^2(x - x_0)}{e^x - 1} dx = \sum_{\text{neb}} \frac{x_k^4}{e^{x_k} - 1} A_k. \tag{58}$$

Thus the temperature of the nebula and the temperature of the nucleus must be found by means of simultaneous solutions of Eqs. (46) and (58).

We have applied this equation to some of the planetary nebulae. The results of the computations are given in Table 13. The first

Table 13. Temperatures of nuclei of some planetary nebulae compared with old values

No.	$\frac{N_2}{H\beta}$	$T_*/1000$		$\frac{T_e}{1000}$	Rad. in cont. spec- trum	Excit. of N_1, N_2	Collis- ions with H atoms	$\frac{\bar{n}_1}{n^+}$	W
		“Nebulium” method	Corr.						
NGC 7672	3.7	59	76	14	10	30	60	0.01	3×10^{-15}
NGC 7009	3.1	40	45	10	15	55	30	0.04	3×10^{-15}
NGC 6572	2.4	40	48	13	15	40	45	0.005	4×10^{-14}
NGC 6826	2.0	27	29	9	25	60	15	0.04	2×10^{-14}
JC II 4593	1.7	24	25	10	30	60	10	0.01	2×10^{-13}
NGC 6543	1.6	33	41	11	20	30	50	0.02	1×10^{-14}

column gives the number of the nebula in the catalog NGC or JC; the second, the ratio $N_2/H\beta$ in accordance with compilation of Berman¹¹; the third, the temperature of the nucleus as determined by Zanstra¹⁰ (method of “nebulium”), Berman,¹² and Page¹³; the fourth, the corrected temperature of the nucleus; the fifth, the electron temperature of the nebula; the sixth, seventh, and eighth, the fractions of the energy of the free electrons expended on captures and free-free transitions, on collisions with atoms of O III and on collisions with hydrogen atoms respectively; the ninth, the quantity n_1/n^+ ; and finally, the tenth, the coefficient of dilution. From Table 13 we can draw the following conclusions:

- (1) The assumption of Zanstra that transformation of the whole

energy of the free electrons into the radiation of the “nebulium” lines appears to be very rough. They lose in this way only half their energy. Therefore, the temperatures of the nuclei determined by the “nebulium” method should be increased by several thousands or tens of thousands of degrees.

(2) The electron temperatures of the planetary nebulae are found to lie in the interval from 9000° to $14,000^\circ$. Strictly speaking, these temperatures are the upper limits of the temperature of nebulae because the electrons can lose their energy in another way, which we have not yet computed. For comparison, we give electron temperatures of nebulae obtained by a different method, namely, from the ratio of intensities of lines $\lambda 4959$ (N_2) and $\lambda 4363$. This method, first pointed out by Ambartsumian,¹⁴ was applied to nebulae by Miyamoto⁹ and by Menzel.¹⁵ The first has obtained temperatures in the interval from $10,000^\circ$ to $25,000^\circ$, and the second, in the interval from $6,000^\circ$ to $10,000^\circ$. The discrepancy of these results is due to the difficulty of the computation of probabilities of the nonelastic collisions of the electrons with atoms of O III.

It is natural to think that the electron temperatures coincide with the temperatures of protons and neutral atoms of hydrogen. However, Zanstra⁶ has pointed out one cause which, in his opinion, leads to very low velocities of the hydrogen atoms in nebulae which are expanding with velocity gradient. This cause is the retardation of the fast-flying atoms during the absorption of $L\alpha$ quanta. Zanstra made approximate computations and obtained the result that each atom during its neutral life should lose almost all its kinetic energy.

As has already been pointed out, however, Zanstra obtained an incorrect value for the density of $L\alpha$ radiation. In addition, he did not take into account the radiation which is emitted by the very fast atoms. Therefore, Zanstra's result is erroneous. More accurate computations (we shall not show them here) indicate that the electron temperatures, and those of protons and neutral atoms of hydrogen are practically identical among themselves. Therefore, the electron temperatures which we found in this section are temperatures of the nebulae as a whole.

We note one more result which comes out of Eq. (37). As is known, the observed Balmer decrement is considerably steeper than

the theoretical one (computed by Cillié). Berman¹¹ has explained an important part of this discrepancy by reddening in space. However, in certain cases, the discrepancy remains, and for this reason Miyamoto proposed a hypothesis of the influence of collisions upon the Balmer decrement. We shall now show that this influence is negligible.

We shall find now the ratio of the number of transitions from the first level to the k th one under the influence of collisions to the number of captures in the k th level. For the whole nebula this ratio is obviously equal to $(D_k/C_k)(\bar{n}_1/n^+)$. Using relation (37) we can write the following series of inequalities:

$$\frac{D_k}{C_k} \cdot \frac{\bar{n}_1}{n^+} < \frac{D_k}{C_k} \frac{\bar{\epsilon} \sum_1^{\infty} C_i}{\sum_2^{\infty} (D_i h\nu_{1i} + D_c h\nu_k)} < \frac{D_k}{D_2} \frac{\sum_1^{\infty} C_i}{C_k} \frac{A_k T_*}{h\nu_{12}} \ll 1. \quad (59)$$

These inequalities establish that collisions of free electrons with hydrogen atoms cannot noticeably change the Balmer decrement which is given by recombination theory (precisely speaking, the energy which is obtained by the electrons owing to photoionization is insufficient for this). As a matter of fact, small discrepancies between theory and observation are apparently to be explained by nonuniformity of the space reddening.

In conclusion, we shall dwell for a while on the methods for the determination of the nuclear temperatures of the nebulae. In addition to the method of "nebulium" mentioned above, Zanstra worked out one more method for the determination of the nuclear temperatures, which for brevity we shall call the method of hydrogen. For the basis of this method one makes only the assumption that the nebula absorbs all the radiation which comes from the star beyond the limit of the Lyman series. Because only one Balmer quantum comes from each L_c quantum which is absorbed by the nebula, this assumption leads to the following relation for the determination of the nuclear temperature:

$$\int_{x_0}^{\infty} \frac{x^2 dx}{e^x - 1} = \sum_H \frac{x_k^3}{e^{x_k} - 1} A_k, \quad (60)$$

where the summation is extended over the Balmer series (including the continuum).

Combining relations (56) and (60), Stoy¹⁶ devised a new method for the determination of the nuclear temperatures, using the relation of the intensities of the “nebulium” lines and the Balmer series of hydrogen. The application of this method to the nebulae and novae (Stoy,¹⁶ Page,¹³ and Oehler¹⁸) has led to considerably lower temperatures than were obtained by the method of Zanstra.

Using the above-mentioned formulae we can now express the following ideas about all methods mentioned above for determination of nuclear temperatures in nebulae:

(1) Making a practical application of the hydrogen method, the greatest difficulties occur in the determination of the quantities A_k for the $H\alpha$ line and for the Balmer continuum. These difficulties we can easily overcome if we take into account Cillié's work. Because the ratio of the number of quanta radiated by the nebulae in the line $H\beta$ to the whole number of Balmer quanta is equal to $n_4 A_{42}/n_e n^+ \sum_2^\infty C_i$, instead of relation (60) we obtain

$$\int_{x_0}^{\infty} \frac{x^2 dx}{e^x - 1} = \frac{n_e n^+ \sum_2^\infty C_i}{n_4 A_{42}} \frac{x_\beta^3}{e^{x_\beta} - 1} A_\beta, \quad (61)$$

where the quantities A_β and x_β refer to the line $H\beta$. An accurate value of T_e is not necessary here in order to determine the nuclear temperature from this relation, because the quantity $n_e n^+ \sum_2^\infty C_i/n_4 A_{42}$ scarcely changes with T_e (one can assume that this quantity is equal to 9.2).

(2) The method of “nebulium” is based on the assumption that all the energy of free electrons is transformed into the radiation of “nebulium” lines. This method obviously gives a lower limit for the nuclear temperature. As has been shown, after refining this method the nuclear temperature should be determined from relation (58), but inserting the quantities A and C from formulae (42) and (44) into this relation (58) we see that it is transformed into relation (61). This means that the refined method of “nebulium” is equivalent to

the method of hydrogen. This result is quite understandable in the physical sense because both methods of Zanstra are based on the comparison of always the same regions of the continuous spectrum of the star (visual region and that beyond the limit of the Lyman series).

(3) If we neglect the first and the third terms on the right-hand side in relation (46) we shall arrive at Eq. (62), which corresponds to the method of Stoy. It is quite understandable that using Eq. (62) for the determination of nuclear temperatures we must get very low values:

$$AT_* = C \frac{N_2}{H\beta}. \quad (62)$$

Thus we come to the following conclusion: the nuclear temperatures of the nebulae should be determined by the method of hydrogen, that is, from Eq. (60) or (61); after this, the temperatures of the nebulae themselves can be found from Eq. (46).

One can, however, suggest another point of view. If the temperatures of the nebulae as determined by Menzel from the ratio of intensities of the lines $\lambda 4959$ and $\lambda 4363$ are correct, then in Eq. (46) the last term is small and this relation takes the form

$$AT_* = BT_e + C \frac{N_2}{H\beta}. \quad (63)$$

The nuclear temperatures as determined from Eq. (63) will, however, be considerably lower than the temperatures determined from Eq. (60). This contradiction can be removed if one makes the assumption that the energy distribution in the spectrum of the nucleus is very different from a Planck distribution. In such case, Eq. (60) gives the color temperature which characterizes the ratio of the intensities in the visual part of the spectrum of the star and beyond the limit of the Lyman series, and Eq. (63) determines the temperature which characterizes the energy distribution beyond the limit of the Lyman series. Relation (46) gives in this case the upper limit of the temperature of the nebula.

IV

ENVELOPES OF NEW STARS

In the present chapter, as in the previous one, we consider the problem of excitation and ionization of atoms in an envelope whose optical depth is, beyond the limit of the fundamental series, greater than 1. In the previous chapter we have considered the very simple case of this problem (the envelope was transparent for the radiation in the lines of subordinate series, and ionization from the excited states does not play a part). Such a case is realized in the envelopes of large radius (planetary nebulae and envelopes of novae in the later stages).

Now we consider a much more complicated case. We shall assume that excitation in the envelope is so great that the envelope is not transparent to the radiation in the lines of subordinate series, and that ionization from excited states plays an important part. This case is realized in envelopes of smaller radius (the envelopes of new stars in the early stage, extended atmospheres of stars which are approximately considered as thin envelopes).

Generally speaking, the problem is reduced to a system of an infinite number of nonlinear integral equations. This system is greatly simplified if one makes the assumption (which is realized in most envelopes) that the optical depth of the envelopes beyond the limits of the subordinate series is less than 1. Even with such an assumption, the system remains too complicated to be solved in general form.

In this chapter we shall proceed as follows. First (Sec. 1) we shall consider an envelope which consists of ideal atoms which possess only three energy state equations (two of them will be discrete and the third corresponds to the ionized state). In this case the problem is solved without difficulty. For a stationary envelope this problem has been solved by Ambartsumian in his well-known paper ("Ionization in the Nebular Envelopes surrounding the Stars."¹). Our

attempt is a generalization of this work for the case of a moving envelope.

Next (Sect. 2) we shall show that the question of excitation and ionization of real atoms which compose the moving envelope is reduced with high approximation to the problem considered in Sec. 1. Next we shall develop the problem and explain the part played by collisions and also by the general absorption in the envelope.

At the end of the chapter some applications of the results will be given, concerning radiation pressure in the envelope, the temperatures of the stars, and so forth.

1. The Atom with Three States

In this section the number of ionized atoms will be denoted by n_3 instead of n^+ , and by the statistical weight of the third state we understand the quantity

$$g_3 = g^+ \frac{(2\pi mkT)^{\frac{3}{2}}}{n_e h^3}. \quad (1)$$

In addition to the width of the spectral line, $\Delta\nu_{12}$, we introduce the effective widths $\Delta\nu_{13}$, $\Delta\nu_{23}$. Then the third state will not be formally different from the first two. The difference between them will be only that for radiation of frequency ν_{12} the Doppler effect will be important, and for radiation of frequency ν_{13} , ν_{23} it will not.

We obtain the condition for radiative equilibrium from the condition of the constancy of numbers of atoms in each state. For the first and third states these conditions will have the form

$$\begin{aligned} n_1 B_{12} \rho_{12} + n_1 B_{13} \rho_{13} &= n_2 A_{21} + n_3 A_{31}, \\ n_1 B_{13} \rho_{13} + n_2 B_{23} \rho_{23} &= n_3 A_{31} + n_3 A_{32}. \end{aligned} \quad (2)$$

Here we neglect the transitions which are caused by collisions, and also the Einstein negative absorption.

Let us introduce quantities which are usually used in the theory of radiative equilibrium. Let α_{ik} be the absorption coefficient per unit volume, J_{ik} the intensity of radiation, and ε_{ik} the amount of

energy radiated per cubic centimeter per second per unit solid angle. Then we have

$$\alpha_{ik} = \frac{n_i B_{ik} h \nu_{ik}}{c \Delta \nu_{ik}}, \quad (3)$$

$$\rho_{ik} = \frac{1}{c} \int J_{ik} d\omega, \quad (4)$$

$$4\pi \varepsilon_{ik} \Delta \nu_{ik} = n_k A_{ki} h \nu_{ik}, \quad (5)$$

and Eqs. (2) can be rewritten in the form

$$\begin{aligned} \frac{\alpha_{12} \Delta \nu_{12}}{h \nu_{12}} \int J_{12} d\omega + \frac{\alpha_{13} \Delta \nu_{13}}{h \nu_{13}} \int J_{13} d\omega &= 4\pi \varepsilon_{12} \frac{\Delta \nu_{12}}{h \nu_{12}} + 4\pi \varepsilon_{13} \frac{\Delta \nu_{13}}{h \nu_{13}}, \\ \frac{\alpha_{13} \Delta \nu_{13}}{h \nu_{13}} \int J_{13} d\omega + \frac{\alpha_{23} \Delta \nu_{23}}{h \nu_{23}} \int J_{23} d\omega &= 4\pi \varepsilon_{13} \frac{\Delta \nu_{13}}{h \nu_{13}} + 4\pi \varepsilon_{23} \frac{\Delta \nu_{23}}{h \nu_{23}}. \end{aligned} \quad (6)$$

It is natural to introduce the quantities K_{ik} and C_{ik} , namely,

$$K_{ik} = J_{ik} \frac{\Delta \nu_{ik}}{h \nu_{ik}}, \quad \alpha_{ik} C_{ik} = \varepsilon_{ik} \frac{\Delta \nu_{ik}}{h \nu_{ik}}. \quad (7)$$

In addition we write

$$\frac{\alpha_{13}}{\alpha_{12}} = \frac{B_{13} \nu_{13} \Delta \nu_{12}}{B_{12} \nu_{12} \Delta \nu_{13}} = q, \quad (8)$$

$$\frac{\alpha_{13} C_{13}}{\alpha_{23} C_{23}} = \frac{A_{31}}{A_{32}} = \frac{p}{1-p}. \quad (9)$$

Then instead of Eqs. (6) we have

$$\begin{aligned} C_{12} &= \int K_{12} \frac{d\omega}{4\pi} - q \left(C_{13} - \int K_{13} \frac{d\omega}{4\pi} \right), \\ C_{13} &= p \int K_{13} \frac{d\omega}{4\pi} + p \frac{\alpha_{23}}{\alpha_{13}} \int K_{23} \frac{d\omega}{4\pi}. \end{aligned} \quad (10)$$

Because the ratio α_{23}/α_{13} is proportional to the quantity C_{12} , the second of these equations is, generally speaking, nonlinear. However, we may assume that the optical depth of the envelope in the frequency ν_{23} is less than 1. Therefore the quantity ρ_{23} can be considered constant.

With the help of the formulae mentioned above, it is easy to find that

$$\frac{\alpha_{23}}{\alpha_{13}} \int K_{23} \frac{d\omega}{4\pi} = \frac{g_3}{g_2} \frac{A_{32}}{A_{21}} \frac{W\bar{\rho}_{23}}{q} \cdot C_{12}, \quad (11)$$

where we use one of the common notations:

$$\sigma_{ik} = \frac{8\pi h\nu_{ik}}{C^3}, \quad \bar{\rho}_{ik} = \frac{1}{e^{h\nu_{ik}/kT} - 1}. \quad (12)$$

We introduce another notation, as follows:

$$\gamma = 3p \frac{g_3}{g_2} \frac{A_{32}}{A_{21}} W\bar{\rho}_{23}. \quad (13)$$

Then, after substituting Eq. (11) in Eqs. (10), we finally obtain

$$\begin{aligned} C_{12} &= \left(1 - \frac{\gamma}{3}\right) \int K_{12} \frac{d\omega}{4\pi} + q(1 - p) \int K_{13} \frac{d\omega}{4\pi}, \\ C_{13} &= p \int K_{13} \frac{d\omega}{4\pi} + \frac{\gamma}{3q} \int K_{12} \frac{d\omega}{4\pi}. \end{aligned} \quad (14)$$

Such are the conditions of radiative equilibrium in our problem.

We introduce the optical distances from the internal boundary of the envelope in the frequencies ν_{12} and ν_{13} :

$$t = \int_{r_1}^r \alpha_{12} d_2, \quad \tau = \int_{r_1}^r \alpha_{13} d_2, \quad (15)$$

where r_1 and r are the distances of the internal boundary of the envelope and the given layer of the star. We denote by θ the angle between the direction of radiation and the external normal to the layer.

Then the equation of radiative transfer in frequency ν_{13} will have the usual form:

$$\cos \theta \frac{dK_{13}}{d\tau} = C_{13} - K_{13}. \quad (16)$$

But for the radiation in frequency ν_{12} , for which the Doppler effect is important, we have obtained a much more complicated equation

in the previous chapter, Eq. (III.18). Assuming that the quantity β is small, we find approximately, instead of Eq. (III.18)

$$K_{12}(t, \theta) = \int_0^t C_{12}(t') e^{-(t-t') \sec \theta (1 + \beta \cos^2 \theta)} \sec \theta dt'. \quad (17)$$

From this we obtain the following equation of radiative transfer for frequency ν_{12} :

$$\cos \theta \frac{dK_{12}}{dt} = - (1 + \beta \cos^2 \theta) K_{12} + C_{12}. \quad (18)$$

Thus our problem reduces to solving Eqs. (14), (16), and (18).

In the previous chapter, while considering the radiative equilibrium in a planetary nebula in frequency ν_{12} , we made an integral equation for the quantity C_{12} . Now, for simplicity, we prefer to obtain the equation of radiative transfer (18) and shall solve this problem by the well-known method of Eddington.

We introduce the following notation:

$$\bar{K}_{ik} = \int K_{ik} \frac{d\omega}{4\pi}, \quad H_{ik} = \int K_{ik} \cos \theta \frac{d\omega}{4\pi}. \quad (19)$$

From Eqs. (16) and (18) we find:

$$\frac{dH_{13}}{d\tau} = -\bar{K}_{13} + C_{13}, \quad \frac{dH_{12}}{dt} = - \left(1 + \frac{\beta}{3}\right) K_{12} + C_{12}, \quad (20)$$

$$\frac{d\bar{K}_{13}}{d\tau} = -3H_{13}, \quad \frac{d\bar{K}_{12}}{dt} = -3H_{12}. \quad (21)$$

(In the last equation we have neglected the quantity β in comparison with 1.) These equations give, with the help of Eq. (14),

$$\begin{aligned} \frac{d^2 \bar{K}_{13}}{d\tau^2} &= 3(1-p)\bar{K}_{13} - \frac{\gamma}{q} \bar{K}_{12}, \\ q^2 \frac{d^2 \bar{K}_{12}}{d\tau^2} &= (\beta + \gamma)\bar{K}_{12} - 3(1-p)q\bar{K}_{13}. \end{aligned} \quad (22)$$

The general form of the solution of the system of Eqs. (22) is

$$\bar{K}_{13} = Ae^{\lambda_1\tau} + Be^{-\lambda_1\tau} + Ce^{\lambda_2\tau} + De^{-\lambda_2\tau}, \quad (23)$$

$$\begin{aligned} \bar{K}_{12} = \frac{q}{\gamma} [3(1-p) - \lambda_1^2](Ae^{\lambda_1\tau} + Be^{-\lambda_1\tau}) \\ + \frac{q}{\gamma} [3(1-p) - \lambda_2^2](Ce^{\lambda_2\tau} + De^{-\lambda_2\tau}), \end{aligned} \quad (24)$$

where λ_1 and λ_2 are the roots of the equation

$$[\lambda^2 - 3(1-p)][q^2\lambda^2 - (\beta + \gamma)] = 3(1-p)\gamma \quad (25)$$

and A , B , C , and D are arbitrary constants.

In what follows we shall assume that

$$\beta \gg q^2. \quad (26)$$

Because the quantity q^2 is very small—of the order of 10^{-8} —this condition is fulfilled for any envelope.

From Eqs. (26) and (25) we obtain

$$\lambda_1 = \left[3(1-p) \frac{\beta}{\beta + \gamma} \right]^{\frac{1}{2}}, \quad (27)$$

$$\lambda_2 = \frac{1}{q} (\beta + \gamma)^{\frac{1}{2}}, \quad (28)$$

and instead of Eq. (24) we have

$$\bar{K}_{12} = \frac{3(1-p)q}{\beta + \gamma} (Ae^{\lambda_1\tau} + Be^{-\lambda_1\tau}) - \frac{\beta + \gamma}{q\gamma} (Ce^{\lambda_2\tau} + De^{-\lambda_2\tau}). \quad (29)$$

To find the arbitrary constants A , B , C , D , we must define boundary conditions. These conditions are defined in the form

$$\begin{aligned} 2H_{12} = \bar{K}_{12}, \quad 2H_{13} = \bar{K}_{13} \quad (\text{for } \tau = \tau_0), \\ 2H_{12} + \bar{K}_{12} = 0, \quad H_{13} = \frac{1}{4}S_{13} \quad (\text{for } \tau = 0). \end{aligned} \quad (30)$$

The quantity πS_{13} , that is, the number of quanta in frequency ν_{13}

which fall from the star on the internal boundary of the envelope, is equal to

$$\pi S_{13} = W \sigma_{13} \bar{\rho}_{13} c \frac{\Delta \nu_{13}}{h \nu_{13}}. \quad (31)$$

The great interest is in the case where the optical depth of the envelope in frequency ν_{13} is considerably larger than 1. ($\tau_0 \gg 1$). In such a case we obtain for the intermediate parts of the envelope

$$\bar{K}_{13} = \frac{3S_{13}}{4\lambda_1} e^{-\lambda_1 \tau}, \quad (32)$$

$$\bar{K}_{12} = \frac{3(1-p)q}{\beta + \gamma} \frac{3S_{13}}{4\lambda_1} e^{-\lambda_1 \tau}$$

(because the remaining members entering into the equation for \bar{K}_{13} and \bar{K}_{12} play only the part of corrections near the limits). Substituting Eqs. (32) in Eqs. (14) for C_{12} and C_{13} , we find

$$C_{12} = \frac{3(1-p)q}{\beta + \gamma} \cdot \frac{3S_{13}}{4\lambda_1} e^{-\lambda_1 \tau}, \quad (33)$$

$$C_{13} = \frac{p\beta + \gamma}{\beta + \gamma} \cdot \frac{3S_{13}}{4\lambda_1} e^{-\lambda_1 \tau}.$$

Knowledge of the quantities C_{12} and C_{13} permits us to find the degree of excitation and ionization in the envelope, that is, the quantities n_2/n_1 and n_3/n_1 . Using the relations (3), (5), and (7) we obtain

$$\frac{n_i}{n_1} = \frac{4\pi}{C} \frac{B_{1i}}{A_{i1}} \frac{h\nu_{1i}}{\Delta\nu_{1i}} C_{1i}, \quad (34)$$

and, inserting Eqs. (33) in Eq. (34), we find

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \bar{\rho}_{12} \frac{3(1-p)}{\lambda_1 p} \frac{\gamma}{\beta + \gamma} e^{-\lambda_1 \tau}, \quad (35)$$

$$\frac{n_3}{n_1} = W \frac{g_3}{g_1} \bar{\rho}_{13} \frac{3}{\lambda_1} \frac{p\beta + \gamma}{\beta + \gamma} e^{-\lambda_1 \tau}.$$

Let us write

$$\beta = z\gamma. \quad (36)$$

Then formulae (35) transform into the forms

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{h\nu_{12}/kT} \frac{1}{p} \left[\frac{3(1-p)}{z(1+z)} \right]^{\frac{1}{2}} e^{-\tau[3(1-p)z/(1+z)]^{\frac{1}{2}}}, \quad (37)$$

$$\frac{n_3}{n_1} n_e = W \frac{g^+}{g_1} \frac{(2\pi mkT)^{\frac{3}{2}}}{h^3} e^{-h\nu_{13}/kT} \frac{3(1+pz)}{[3(1-p)z(1+z)]^{\frac{1}{2}}} e^{-\tau[3(1-p)z/(1+z)]^{\frac{1}{2}}}. \quad (38)$$

These formulae appear to us to be final.

We note that in the case of a stationary envelope, that is, where $\beta = 0$, the quantities \bar{K}_{12} and \bar{K}_{13} , and therefore the quantities n_2/n_1 and n_3/n_1 , are not exponential functions, as they are here, but linear functions of the optical distance.

Up to the present we had thought that in a nebular envelope, the degree of excitation and ionization depended on three quantities: temperature of the star, dilution coefficient, and density. Now we see that it depends also (and very strongly) on the condition of motion of the envelope. The larger is the velocity gradient in the envelope, the less is the degree of excitation and ionization, and the faster it diminishes in the transition from the internal to external boundary of the envelope.

2. The Real Atom

We shall now consider the real atom, which possesses an infinite number of levels. The condition of stationarity for the i th level has the form

$$\sum_{k=1}^{i-1} (n_i A_{ik} - n_k B_{ki} \rho_{ki}) + n_i B_{ic} \rho_{ic} = \sum_{k=i+1}^{\infty} (n_k A_{ki} - n_i B_{ik} \rho_{ik}) + n_e n^+ C_i. \quad (39)$$

Utilizing the notations introduced in the preceding section we may write

$$\begin{aligned} n_k A_{ki} - n_i B_{ik} \rho_{ik} &= 4\pi \alpha_{ik} (C_{ik} - \bar{K}_{ik}), \\ n_e n^+ C_i - n_i B_{ic} \rho_{ic} &= 4\pi \alpha_{ic} (C_{ic} - \bar{K}_{ic}). \end{aligned} \quad (40)$$

Therefore instead of Eq. (39) we obtain

$$\sum_{k=1}^{i-1} \alpha_{ki} (C_{ki} - \bar{K}_{ki}) = \sum_{k=i+1}^{\infty} \alpha_{ik} (C_{ik} - \bar{K}_{ik}) + \alpha_{ic} (C_{ic} - \bar{K}_{ic}). \quad (41)$$

To these conditions of radiative equilibrium we must add also an equation of radiative transfer. For the radiation in the continuous spectrum we have, as always,

$$\cos \theta \frac{dK_{ic}}{dt_{ic}} = -K_{ic} + C_{ic}, \quad (42)$$

and for the radiation in the spectral lines, by analogy with Eq. (18) of the previous section, we have

$$\cos \theta \frac{dK_{ik}}{dt_{ik}} = -(1 + \beta_{ik} \cos^2 \theta)K_{ik} + C_{ik}, \quad (43)$$

where

$$\beta_{ik} = \frac{1}{2u} \cdot \frac{dv}{dt_{ik}}. \quad (44)$$

The solution of the system of equations (41), (42), and (43) presents great difficulties. For the stationary envelope, that is, for $\beta_{ik} = 0$, this system has been considered by Ambartsumian² and Henyey.³ They obtained several first integrals for the system (flux integrals). It is easy to see that in the present case, that is, with velocity gradient, the flux integrals do not exist.

The presence of a velocity gradient leads, however, to an essential simplification of the above-mentioned system. If the velocity gradient is sufficiently large, then for each volume element of the internal parts of the envelope the following condition can be assumed to be fulfilled: the number of quanta radiated by a spectral line is equal to the number of quanta absorbed by the same line, and the number of quanta which leave the envelope owing to the Doppler effect. In the case of plane parallel layers, this condition has the form

$$C_{ik} = \bar{K}_{ik} + \frac{\beta_{ik}}{3} \bar{K}_{ik}. \quad (45)$$

This circumstance, that the condition should actually be fulfilled, follows from the results obtained in the previous section. Actually, assuming that the condition (26) is fulfilled, we have obtained the result that in the average parts of the envelope the radiation density is determined from formulae (32) (in addition, as is easy to see,

this result does not depend on the boundary conditions for line radiation). But we should have come to the same formula (32) if, together with Eqs. (14) and (16), we had considered not Eq. (18) of radiative transfer, but Eq. (45) for the frequency ν_{12} . This means that with the fulfillment of the inequality (26) relation (45) can also be assumed to be fulfilled.

Taking Eq. (45) and assuming, as previously, that the optical depth of the envelope beyond the limits of subordinate series is less than unity (then the quantities ρ_{2c} and ρ_{3c} , and so forth, will be given), we see that the problem consists essentially in finding the density of radiation beyond the limit of the fundamental series (because if all the quantities ρ_{ic} are known, then the computation of the degree of excitation at each place in the envelope by means of Eq. (45) represents an algebraic operation). Certainly in reality the quantity ρ_{ic} depends itself on the degree of excitation, but we shall show presently that for approximate values of this quantity it is sufficient to draw attention only to the two first levels.

Adding term by term all of Eq. (41), beginning from the first term, we obtain

$$\sum_{i=2}^{\infty} \alpha_{1i}(C_{1i} - \bar{K}_{1i}) = \sum_{i=2}^{\infty} \alpha_{ic}(C_{ic} - \bar{K}_{ic}). \quad (46)$$

But

$$\alpha_{ic}C_{ic} = \frac{p_i}{p_1} \alpha_{1c}C_{1c}, \quad (47)$$

where p_i is the fraction of the captures on the i th level, and

$$\alpha_{ic}\bar{K}_{ic} = \alpha_{1c} \frac{B_{ic}\rho_{ic}}{q_i A_{i1}} \bar{K}_{1i}, \quad (48)$$

where $q_i = \alpha_{ic}/\alpha_{1i}$. In addition we may use Eq. (45) for lines of the fundamental series. Then instead of Eq. (46) we find

$$\frac{\beta_{12}}{3} \sum_{i=2}^{\infty} \bar{K}_{1i} = q_2 \frac{1-p_1}{p_1} C_{1c} - \sum_{i=2}^{\infty} \frac{q_2}{q_i} \frac{B_{ic}\rho_{ic}}{A_{i1}} \bar{K}_{1i}. \quad (49)$$

But the first equation (41) reduces to the form

$$\frac{\beta_{12}}{3} \sum_2^{\infty} K_{1i} + q_2(C_{1c} - \bar{K}_{1c}) = 0. \quad (50)$$

It is easy to obtain the following relation:

$$\frac{\sum_{i=2}^{\infty} (q_2/q_i)(B_{ic}\rho_{ic}/A_{i1})\bar{K}_{1i}}{\sum_{i=2}^{\infty} \bar{K}_{1i}} = \frac{B_{2c}\rho_{2c}}{A_{21}} \frac{\sum_{i=2}^{\infty} (B_{1c}\rho_{1c}/B_{2c}\rho_{2c})n_i}{\sum_{i=2}^{\infty} (\sigma_{1i}/\sigma_{12})(g_2/g_i)n_i} \simeq \frac{B_{2c}\rho_{2c}}{A_{21}} \quad (51)$$

(because in both sums the chief part is played by the first terms). Therefore from Eqs. (49) and (50) we obtain

$$\left(\beta_{12} + 3p_1 \frac{B_{2c}\rho_{2c}}{A_{21}} \right) C_{1c} = \left(\beta_{12} + 3 \frac{B_{2c}\rho_{2c}}{A_{21}} \right) p_1 \bar{K}_{1c}. \quad (52)$$

This equation, together with the corresponding equation of radiative transfer (42) does solve the problem of ionization in the envelope. But it is easy to see that Eq. (52) could have been obtained if we had taken into account only the first two levels. Consequently the problem of excitation and ionization of the real atom actually reduces to the problem which we considered in the previous section.

3. Collisions and General Absorption in the Envelope

The presence of a velocity gradient is not the only unique cause which leads to the decrease of degree of excitation and ionization of atoms. The same influence is produced by collisions of the second kind and general absorption in the envelope. We shall now consider each of these separately.

(a) *Collisions*. If there is a discrepancy between the degree of excitation of atoms and the electron temperature, then collisions between atoms and electrons tend to diminish it. If the degree of excitation is below that of Boltzmann at the temperature of the electron gas, then collisions of the first kind occur more often than those of the second kind and lead to an increase of the degree of excitation and to a decrease of the electron temperature. In the reverse situation, collisions of the second kind occur more often, and this leads to a diminution of the degree of excitation and to an increase of the electron temperature. In the gaseous nebulae, the first of these cases is realized; in envelopes of smaller radius, apparently both cases may be realized.

We shall assume that we have the second case and shall explain in more detail the importance of collisions. As in Sec. 1, we shall treat the three-state atom. In order to take into account collisions of the second kind, we should add the term $n_2 n_e D_{21}$ to the right-hand side of the first equation in (2). Then instead of the first relation (14) we obtain

$$C_{12} = \left(1 - \frac{\gamma}{3}\right) \bar{K}_{12} + q(1 - p) \bar{K}_{13} - \frac{n_2 n_e D_{21}}{4\pi\alpha_{12}}. \quad (53)$$

But

$$\frac{n_2}{4\pi\alpha_{12}} = \frac{n_e D_{21}}{A_{21}}. \quad (54)$$

Therefore, arriving at Eqs. (22) we find, instead of the second of these equations,

$$q^2 \frac{d^2 \bar{K}_{12}}{d\tau^2} = (\beta + \gamma + \delta) \bar{K}_{12} - 3(1 - p)q \bar{K}_{13}, \quad (55)$$

where we put

$$\frac{\delta}{3} = \frac{n_e D_{21}}{A_{21}}. \quad (56)$$

We see that the influence of collisions of the second kind is manifested formally in the same way as the presence of a velocity gradient.

For the estimation of the quantity δ we can avail ourselves of Eq. (III.49), which determines the quantities D_{1k} for the hydrogen atom. Because

$$D_{21} = \frac{g_1}{g_2} e^{h\nu_{12}/kT} D_{12}, \quad (57)$$

we find at $T = 10,000^\circ$ (the quantity D_{21} , incidentally, depends only slightly on the temperature):

$$\delta \approx 10^{-17} n_e. \quad (58)$$

For the stellar envelopes considered in Chapter II we obtained $\beta > 10^{-3}$ and $n_e < 10^{12}$. This means that in such envelopes, $\beta \gg \delta$, that is, the velocity gradient plays a much larger part than collisions of the second kind. In the envelopes of novae the situation appears to be analogous.

(b) *General Absorption in the Envelope.* In the presence of general absorption in the envelope the decrease of degree of excitation and ionization is due to the absorption of quanta which are radiated by the atoms present beyond the series limits as well as to the absorption of quanta radiated in the spectral lines. In order to investigate this effect we must add the corresponding terms in the equations of radiative transfer written above. (The condition of radiative equilibrium is obviously unchanged by this.)

Let us take again a three-state atom. Instead of Eqs. (16) and (18) we now have

$$\cos \theta \frac{dK_{13}}{d\tau} = -(1 + \eta)K_{13} + C_{13}, \quad (59)$$

$$\cos \theta \frac{dK_{12}}{dt} = -(1 + \eta_1 + \beta \cos^2 \theta)K_{12} + C_{12}, \quad (60)$$

where

$$\eta = \frac{\alpha'_{13}}{\alpha_{13}}, \quad \eta_1 = \frac{\alpha'_{12}}{\alpha_{12}}, \quad (61)$$

and α'_{13} and α'_{12} are coefficients of total absorption beyond the limit of the fundamental series and in the spectral lines.

From Eqs. (59) and (60) and by means of the condition of radiative equilibrium we obtain

$$\begin{aligned} \frac{d^2 \bar{K}_{13}}{d\tau^2} &= 3(1 - p + \eta)\bar{K}_{13} - \frac{\gamma}{g}\bar{K}_{12}, \\ q^2 \frac{d^2 \bar{K}_{12}}{d\tau^2} &= (3\eta_1 + \beta + \gamma)\bar{K}_{12} - 3q(1 - p)\bar{K}_{13}, \end{aligned} \quad (62)$$

instead of Eqs. (22). The characteristic equation of this system has the form

$$[\lambda^2 - 3(1 - p + \eta)][q^2 \lambda^2 - (3\eta_1 + \beta + \gamma)] = 3(1 - p)\gamma, \quad (63)$$

and, if the inequality (26) is assumed to be fulfilled, then for λ_1 we find

$$\lambda_1^2 = 3 \left[\eta + (1 - p) \frac{3\eta_1 + \beta}{3\eta_1 + \beta + \gamma} \right]. \quad (64)$$

It is difficult to say, not having data on actual envelopes and

atoms, which of the terms that enter this formula plays the most important part. We remark only that in the envelopes of stars of later classes the part played by the total absorption is very great. Therefore this equation will be considered in detail in the next chapter.

4. Certain Applications

There is no doubt that the formulae derived by us should be taken into consideration in the solution of many problems relating to the gaseous envelopes around stars. Certain of these questions are discussed below.

(a) *Nonhomogeneous Envelopes.* In Chapter II we applied the results obtained in Chapter I for a homogeneous medium to the nonhomogeneous envelopes of stars. At that time we could not justify the legitimacy of such a procedure because we did not possess the equation of radiative transfer in the spectral lines. We now do this, taking the three-state atom for simplicity.

In the given case, the radiation density in a line is determined by the second of Eqs. (22), in which the quantities β , γ , and K_{13} are prescribed functions of r . Our approximation consists in assuming that, in the mean path through the envelope, the radiation density in the line is

$$\bar{K}_{12} = \frac{3(1-p)q}{\beta + \gamma} \bar{K}_{13}. \quad (65)$$

From this equation follows the inequality which gives a measure of the conditions under which this approximation is justified:

$$\frac{d^2}{dt^2} \left(\frac{\bar{K}_{13}}{\beta + \gamma} \right) \ll \bar{K}_{13}. \quad (66)$$

In the model of the envelope already considered in Chapter II, we have $\bar{K}_{13} \sim (r_0/r)^2$, $dt = \alpha 12^0 (r_0/r)^2 dr$, $\beta = (v/3u)(1/\alpha 12^0)(r/r_0)$. We shall assume that $\beta \gg \gamma$, because otherwise the distribution of atom states in the whole envelope will be near to that of Boltzmann.

For such an envelope, the inequality (66) is reduced to

$$\frac{18}{\alpha_{12}^0 r_0} \frac{u}{v} \cdot \frac{r}{r_0} \ll 1. \quad (67)$$

Since usually $\alpha_{12}^0 r_0 \sim 10^4$, $u/v \approx 0.01$, this inequality can be

considered as fulfilled to a very high degree over the whole extension of the envelope.

(b) *Temperatures of Stars.* As is known, the method of Zanstra for the determination of the temperatures of stars surrounded by nebulous envelopes is based on the assumption that the luminosity of the envelope in the lines of some element is produced at the expense of the energy of the stars beyond the limit of the fundamental series. In envelopes of small radius, however, such a transformation of energy occurs not only beyond the limit of the fundamental series, but also beyond the limits of the subordinate series. Therefore the temperatures of stars as determined by the Zanstra method without taking this circumstance into account may appear to be too high.

Gordeladze⁴ estimated the magnitude of this effect, using the theory of excitation and ionization in envelopes of smaller radius as given by Ambartsumian.¹ He found that for the WR stars the temperatures determined from the lines of He II and other atoms with high ionization potential are much higher than those obtained from the hydrogen lines, and therefore should be considerably smaller if this effect is taken into account.

The theory of Ambartsumian does not, however, take into account the motion of the envelope. We shall now estimate the magnitude of this effect, using the formulae given above.

Let us assume that the envelope absorbs all the energy of the star beyond the limit of the fundamental series. The number of absorbing quanta will be

$$N_1 = \frac{8\pi^2 r_*^2}{c^2} \int_{\nu_{13}}^{\infty} \frac{\nu^2 d\nu}{e^{h\nu/kT} - 1} = \frac{8\pi^2 r_*^2}{c^2} \nu_{13}^2 \Delta\nu_{13} \bar{\rho}_{13}. \quad (68)$$

We now find the number of quanta beyond the limits of the subordinate series absorbed by the envelope. Obviously we have

$$N_2 = 4\pi r_1^2 \int_{r_1}^{r_2} n_2 B_{23} \rho_{23} dr = 4\pi r_1^2 \frac{B_{23} \rho_{23} C \Delta\nu_{13}}{B_{13} h \nu_{13}} \int_0^{\infty} \frac{n_2}{n_1} d\tau, \quad (69)$$

or, using the first of Eqs. (35),

$$N_2 = 4\pi r_1^2 \frac{B_{23} \rho_{23} C \Delta\nu_{13}}{p B_{13} h \nu_{13}} \frac{g_2}{g_1} \bar{\rho}_{12} \frac{\gamma}{\beta}. \quad (70)$$

For the ratio N_2/N_1 we have

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} \frac{B_{23}}{B_{13}} \frac{\bar{\rho}_{12}\bar{\rho}_{23}}{p\bar{\rho}_{13}} \left(\frac{\nu_{23}}{\nu_{13}}\right)^3 \frac{\gamma}{\beta} \simeq \frac{\gamma}{\beta} \quad (71)$$

This expression should be inserted into the following equation:

$$\left(1 + \frac{N_2}{N_1}\right) \int_{x_0}^{\infty} \frac{x^2 dx}{e^x - 1} = \sum \frac{x_k^3}{e_k^x - 1} A_k, \quad (72)$$

which is the general form of Zanstra's equation (III.60). Here the summation is carried out over the lines of the atom considered.

Computation shows that the quantity γ/β is usually about unity. Therefore we arrive at the conclusion that the effect under consideration does not play an important part (for ionized helium this quantity should be of the order of 10^8 in order that the temperature of the star should be "increased" from $20,000^\circ$ to $50,000^\circ$).

The real explanation of the discrepancy mentioned above between the temperature determined for different atoms is given in Sec. 1 of Chapter II.

(c) *Temperatures of the Envelopes.* The problem of the temperatures of envelopes of small radius (in particular, the envelopes of the novae) is at the present time completely unsolved. Usually we assume that the temperature of the envelope is simply equal to that of the star. To solve the problem we may, in principle, use the method proposed by us for the determination of temperatures of gaseous nebulae (Chapter III, Sec. 3). Owing to the complication of the phenomena occurring in envelopes of small radius, however, accurate computations are still difficult.

Nevertheless, we can note one circumstance which makes an envelope of small radius different from that of a gaseous nebula. Because the ionization in this envelope originates not only from the ground level but also from the excited levels, the average energy received by electrons from ionization is in this case considerably smaller than in the nebulae. As an illustration we make a computation for the hydrogen envelope, neglecting collisions.

The condition for equality of number of ionizations and re-combinations and the law of conservation of energy for the free electrons of the envelope may be written in the form

$$\sum_1^{\infty} n_i B_{ic} \rho_{ic} = n_e n^+ \sum_1^{\infty} C_i, \quad (73)$$

$$\sum_1^{\infty} n_i B_{ic} \rho_{ic} \varepsilon_{ic} = n_e n^+ \left(\sum_1^{\infty} C_i \varepsilon_i + f \right), \quad (74)$$

where ε_{ic} is the average energy received by an electron from photo-ionization from the i th level, and the remaining notation is the same as in Sec. 3 of Chapter III.

We assume as before that the optical depth of the envelope beyond the limit of the fundamental series is much greater than unity, and that beyond the limits of subordinate series it is less than unity. Then all quanta radiated by the envelope beyond the limit of the fundamental series will be absorbed in the envelope itself. Therefore instead of Eqs. (73) and (74) we obtain

$$\sum_1^{\infty} n_i B_{ic} \rho_{ic}^0 = n_e n^+ \sum_2^{\infty} C_i, \quad (75)$$

$$\sum_1^{\infty} n_i B_{ic} \rho_{ic}^0 \varepsilon_{ic}^0 = n_e n^+ \left(\sum_2^{\infty} C_i \varepsilon_i + f \right), \quad (76)$$

where by ρ_{ic}^0 we denote the radiation density coming directly from the star, and by ε_{ic}^0 the corresponding energy which the electrons receive by ionization.

Let N_i be the number of ionizations occurring from the i th state in the whole envelope, that is,

$$N_i = \int n_i B_{ic} \rho_{ic}^0 dV. \quad (77)$$

From relations (75) and (76) we find:

$$\frac{\varepsilon_{1c}^0 + (N_2/N_1)\varepsilon_{2c}^0 + \dots}{1 + N_2/N_1 + \dots} = \frac{\sum_2^{\infty} C_i \varepsilon_i + f}{\sum_2^{\infty} C_i}. \quad (78)$$

As in Chapter III, this relation can be transformed into

$$\frac{A_1 + (N_2/N_1)A_2 + \dots}{1 + N_2/N_1 + \dots} T_* = BT_e, \quad (79)$$

where the quantity A_1 is determined by Eq. (III.42), and the quantities A_2, A_3, \dots by Eq. (III.41), in which we should set $x_0 = h\nu_{ic}/kT$ (because beyond the limit of the fundamental series all radiation coming from the star is absorbed, but beyond the limits of subordinate series, only a small part of it). Equation (79) gives the required relation between T_* and T_e .

As an example we assume that the majority of ionizations are from the second level. Then instead of Eq. (79) we have

$$A_2 T_* = BT_e. \quad (80)$$

With $T_* = 20,000^\circ$, from Table 11 we find $A_2 = 0.71$ (this value is taken from the first part of the table, with a temperature four times the temperature of the star). With this value of A_2 we find the electron temperature, T_e , to be $12,000^\circ$. We note that if N_2/N_1 were less than unity, T_e would be of the order of $20,000^\circ$.

As has been mentioned, the result just obtained does not pretend to accuracy, because we have not taken collisions into account, which can decrease as well as increase the temperature of the envelope (corresponding to collisions of the first and second kinds).

We should not forget also that the temperature in different parts of the envelope may be different. In such a case, the temperature of the outer parts must be lower than that of the inner parts. This is due to two causes: first, the decrease of degree of excitation and ionization during the transition from the internal to the external boundary, and second, the total occultation of the radiation of the star beyond the limit of the fundamental series. Consequently, the electrons receive from ionization a small fraction of the energy and cannot increase it because of collisions of the second kind.

(d) *Radiation Pressure.* There is no doubt that radiation pressure caused by the absorption of light in the spectral lines plays an important part in the envelopes of stars. In the application it is important to give the formulae that determine the following

quantities: first, light pressure at the internal boundary of the envelope, and second, light pressure affecting the whole envelope.

In order to obtain these formulae we can use the results which were found in Sec. 1 of the present chapter. After determining the arbitrary constants entering Eqs. (23) and (24) from the boundary condition (30), we find the flux of radiation at the internal boundary of the envelope:

$$4\pi H_{12}(0) = -q \sqrt{\frac{3(1-p)}{\beta}} \pi S_{13}. \quad (81)$$

The radiation pressure computed by means of this formula appears to be of an order that is comparable to the gravitational force of the central star.

In deriving Eq. (81) we neglected the line radiation which falls on the internal boundary of the envelope and comes from the star. If we take this into account, we should add the following term to the flux of radiation (81):

$$4\pi H'_{12}(0) = + \beta^{\frac{1}{2}} \pi S_{12}. \quad (82)$$

We should consider that this term is much smaller than the preceding one, because the ratio of S_{13}/S_{12} , that is, the ratio of the quantum number beyond the limit of the fundamental series to the quantum number in the line is understood to be very great.

We note that in an explanation of the phenomena occurring in the envelopes of novae immediately after maximum, Mustel^{5, 6} assumed that the major force which operates on the envelope during this period is the light pressure caused by the absorption of the quanta which come directly from the star. The radiation flux at the internal boundary of the envelope is, on this assumption,

$$4\pi H''_{12}(0) = + \pi S_{12}. \quad (83)$$

In order to bring this assumption and our results into agreement, we should require the fulfillment of one of two conditions: (1) the velocity gradient in the envelope is very large, but the energy of the star beyond the limit of the fundamental series is insufficient to produce strong fluorescence [then $|H_{12}(0)| < H'_{12}(0)$ and $H'_{12}(0) \approx H''_{12}(0)$]; this means, however, that one should find another

mechanism (not fluorescence) to explain the bright lines observed in the spectra of novae at this time; (2) radiative equilibrium is not present in the envelope (see below).

In finding the light pressure that operates on the whole envelope, McCrea⁷ and Mustel⁸ assumed that the flux in the lines does not change in the envelope (that is, they accepted the Schuster model). In reality, one should take fluorescence into account as well as the presence of a velocity gradient. The corresponding formulae can easily be derived by means of the results obtained above; we shall not, however, dwell on this point.

(e) *Relaxation Time.* For the envelopes of new stars it is important to determine the relaxation time, that is, the interval that is necessary for the establishment of radiative equilibrium. The results of the theory of radiative equilibrium can be applied only to those envelopes whose duration is great compared to the time of relaxation.

We may consider that the relaxation time is determined by the time during which the light quantum is found in the envelope. Therefore for the calculation of the relaxation time one should divide the number of quanta found in the envelope by the number of quanta in a column of 1 cm² cross section; we obtain

$$\frac{4\pi}{c} \int_{r_1}^{r_2} (\bar{K}_{12} + \bar{K}_{13}) dr = \frac{3}{\beta} \frac{\pi S_{13}}{c \alpha_{13}} \left[q + \frac{\beta + \gamma}{3(1 - p)} \right]. \quad (84)$$

Because the number of quanta leaving this column in 1 sec is obviously equal to πS_{13} , we find the time of relaxation

$$T = \frac{6u}{c \, dv/dr} \left[1 + \frac{\beta + \gamma}{3q(1 - p)} \right]. \quad (85)$$

In the initial stage of the envelopes of new stars we can assume $dv/dr \approx v/r$ and $\beta + \gamma \approx q$. In such a case, Eq. (77) gives for the time of relaxation a quantity of the order of several hours.

This interval of time is certainly too small for the astronomical processes. However, in the envelopes of new stars such rapid changes occur immediately after maximum that the assumption of the absence of radiative equilibrium at this time does not appear unreasonable.

V

STARS OF LATE CLASS WITH BRIGHT LINES

The most complicated case appears to be the envelopes whose optical depth is large in comparison with unity, not only beyond the limit of the fundamental series, but also beyond the limits of subordinate series. Because the motion of the envelope does not affect the radiation in the continuous spectrum, in this case we cannot use the advantages which are given by the presence of a velocity gradient. We do not intend to make a detailed discussion of this case now. However we shall show—and this is the basic purpose of the present chapter—that investigation of such envelopes leads to the explanation of one of the most puzzling questions of modern astrophysics—the origin and behavior of bright lines in the spectra of stars of late type.

As is well known, stars of late class with bright lines include long-period variables, stars of Z Andromedae type, and certain other groups. Long-period variables exhibit in their spectra bright lines of hydrogen and ionized iron (near maximum brightness) and neutral iron (near minimum brightness). In the spectra of Z Andromedae type stars, bright lines of hydrogen, helium, ionized helium, and other atoms with very high ionization potential are observed. In the spectra of many of these stars forbidden lines, characteristic of the gaseous nebulae, are also seen.

For an explanation of the origin of bright lines in the spectra of stars of late type, several hypotheses have been proposed. The first of these, stated first by Russell¹ and developed by Wurm,² sees the cause of the appearance of the bright lines in the process of chemiluminescence. This explanation can, however, only have importance for the few lines with low excitation potential. According to another hypothesis, the stars with “combination spectra” are in reality double, consisting of a cool and a hot component. The relative

motions of these components and the pulsations of their atmospheres may be used to explain the observed changes of brightness and spectrum. This hypothesis was first put forward by Berman³ for the star R Aquarii. Nowadays it appears to be universally accepted for all stars of Z Andromedae type. Finally, according to a third hypothesis, the origin of bright lines is due to fluorescence, that is, they result from photoionization and recombination. Until the present time, however, it has not been shown whence comes the powerful high-frequency radiation necessary for the photoionization if the temperature of the reversing layer of the star is so small (of the order of 2000° or 3000°). According to the idea recently expressed by Shajn,⁴ this radiation is not associated with radiative equilibrium in the external layers of the star.

No doubt the third of these hypotheses appears to be the most acceptable. Observation definitely indicates that the bright lines in the spectra of stars of later classes originate by means of the same mechanism which occurs in the spectra of stars of early classes, that is, as a result of fluorescence (see Sec. 3). Therefore there arises the problem of giving a theoretical interpretation of this phenomenon. The solution of this problem is our concern in this chapter. We assume here that radiative equilibrium exists in the internal layers of the star. We reject the assumption of local thermodynamic equilibrium, however.

Up to the present, different authors have attempted to explain the origin of bright lines in the spectra of late class. We reverse the problem. We consider the same model that has been introduced in the preceding chapters: a hot star surrounded by an envelope which is located at some distance from the star. It is clear that the envelope will give emission lines which originate by fluorescence, and the higher the temperature of the star, the higher will be the ionization potential of the radiating atoms. Our problem will consist in showing that, if certain conditions are fulfilled, the envelope should give a line and continuous spectrum of later type, as well as the emission lines.

In Sec. 1 of this chapter, we give the essential explanation of the origin of "combination spectra." In Sec. 2 we refine this, and in Sec. 3 we give a series of general ideas respecting the spectra considered.

1. Origin of "Combination Spectra"

We shall consider an envelope surrounding a hot star, and shall assume that it is in motion. In the beginning we shall apply to this envelope the results obtained in the previous chapter. We shall compute and intercompare the following quantities: (1) the energy radiated by the star in the continuum; (2) the energy radiated by the envelope in the continuum; (3) the energy radiated by the envelope in spectral lines. The calculations will refer to a hydrogen envelope.

If the envelope is transparent to the radiation of subordinate series and absorbs only the radiation from the star beyond the limit of the fundamental series, then, as is well known, the continuous spectrum of the envelope in the observed part will be very weak in comparison with the continuous spectrum of the star. At the same time, however, the bright lines will be very strong on the background of the continuous spectrum of the envelope. Such a situation is found in the gaseous nebulae. It is different in envelopes of small radius. These envelopes absorb the radiation coming from the star, not only beyond the limit of the fundamental series but also beyond the limits of subordinate series. On this account, the intensity of the continuous spectrum of the envelope increases greatly. The intensity of the bright lines, however, does not undergo such a great increase because of the nontransparency of the envelope to line radiation. Therefore, with considerable nontransparency of the envelope to line radiation (more accurately speaking, for small values of the parameter x , introduced previously), we should expect a rather strong continuous spectrum from the envelope in comparison with the continuous spectrum of the star, with comparative weakness of bright lines. We shall now verify this conclusion by direct calculations.

Let E_ν^* be the energy radiated by the star in frequency ν , and let E_ν be the energy radiated by the envelope in the same frequency. These quantities are determined by

$$E_\nu^* = 4\pi r_*^2 \pi \frac{2h\nu^3}{c^2} \cdot e^{\frac{1}{hv/kT_*}} - 1, \quad (1)$$

$$E_\nu = 4\pi r_1^2 \int_{r_1}^{r_2} n_e n^+ \epsilon_\nu dr, \quad (2)$$

where

$$\varepsilon_\nu = \frac{2^7 \pi^3}{(6\pi)^{\frac{3}{2}}} \left(\frac{m}{kT_e} \right)^{\frac{1}{2}} \frac{e^6}{c^3 m^2} e^{-h\nu/kT_e} \left(1 + 2 \frac{h\nu_0}{kT_e} \sum_{i=3}^{\infty} \frac{1}{i^3} e^{xi/kT_e} \right). \quad (3)$$

In (3) the first term in parentheses takes into account free-free transitions and the second, free-bound transitions. The summation is carried from $i = 3$ because E_ν is to be computed for the visual part of the spectrum.

Let E_{k2} be the total energy radiated by the envelope in the Balmer lines, which correspond to the transition $k \rightarrow 2$. This quantity is

$$E_{k2} = 4\pi r_1^2 A_{k2} h\nu_{2k} \int_{r_1}^{r_2} n_k \frac{\beta_{2k}}{3} dr, \quad (4)$$

where $\beta_{2k}/3$ is the fraction of the quanta leaving the envelope in the given line owing to the Doppler effect.

For the calculation of integrals (2) and (4) we use the formulae that determine the degree of excitation and ionization in the envelope, as derived in the previous chapter. These formulae have the form

$$\begin{aligned} \frac{n_2}{n_1} &= \frac{g_2}{g_1} e^{-h\nu_{12}/kT_*} \frac{1}{p} \left[\frac{3(1-p)}{z(1+z)} \right]^{\frac{1}{2}} e^{-\tau[3(1-p)z/(1+z)]^{\frac{1}{2}}}, \\ n_e \frac{n^+}{n_1} &= W \frac{g^+}{g_1} \frac{(2\pi mkT_*)^{\frac{3}{2}}}{h^3} e^{-h\nu_0/kT_*} \frac{3(1+pz)}{[3(1-p)z(1+z)]^{\frac{1}{2}}} \\ &\quad \times e^{-\tau[3(1-p)z/(1+z)]^{\frac{1}{2}}}, \end{aligned} \quad (5)$$

where the quantity z is connected with x by the relation

$$z = \frac{x A_{21}}{p B_{2c} \rho_{2c}^*}, \quad (6)$$

and x is given by

$$x = \frac{\beta_{12}}{3W}. \quad (7)$$

We recall that in deriving Eq. (5) the quantity z was assumed to be constant in the envelope.

Instead of Eq. (2) we can write

$$E_\nu = \pi r_*^2 \varepsilon_\nu \int_0^\infty \frac{n_e n^+}{W n_1 \alpha_{1c}} d\tau, \quad (8)$$

where τ is the optical depth beyond the limit of the fundamental series. Inserting the second of Eqs. (5) into Eq. (8), we obtain

$$E_\nu = \pi r_*^2 \frac{4\pi h\nu^3}{c^2} \cdot \frac{kT_*}{h\nu_0} e^{-h\nu_0/kT_*} \left(1 + 2 \frac{h\nu_0}{kT_e} \sum_3 \frac{e^{x_i/kT_e}}{i^3} \right) \times e^{-h\nu/kT_e} \left(1 + \frac{B_{2c}\rho_{2c}^*}{xA_{21}} \right). \quad (9)$$

From this we see that E_ν increases rapidly with decreasing x .

To calculate the integral (4) we must split it into two integrals, one over the region nontransparent to radiation in the given line, and the other over the region transparent to this radiation. Doing this we find for E_{k2} :

$$E_{k2} = 4\pi r_1^2 A_{k2} h\nu_{2k} \left[\int_0^{\tau_1} n_k \frac{\beta_{2k}}{3} \frac{d\tau}{\alpha_{1c}} + \int_{\tau_1}^{\infty} n_k \frac{d\tau}{\alpha_{1c}} \right]. \quad (10)$$

This formula can be transformed into

$$E_{k2} = \pi r_*^2 A_{k2} h\nu_{2k} \left[\frac{n_k}{n_2} \times \frac{\alpha_{12}}{\alpha_{2k}} \int_0^{\tau_1} \frac{d\tau}{\alpha_{1c}} + \frac{n_k}{n_e n^+} \int_{\tau_1}^{\infty} \frac{n_e n^+}{W n_1} \frac{d\tau}{\alpha_{1c}} \right]. \quad (11)$$

Integrating, and using the fact that, for $\tau = \tau_1$,

$$\frac{n_k}{n_2} \times \frac{\alpha_{12}}{\alpha_{2k}} = \frac{n_k}{n_e n^+} \cdot \frac{n_e n^+}{W n_1}, \quad (12)$$

we write instead of Eq. (11):

$$E_{k2} = \pi r_*^2 \frac{A_{k2} h\nu_{2k}}{\alpha_{1c}} \cdot \frac{n_k}{n_2} \times \frac{\alpha_{12}}{\alpha_{2k}} \left\{ \tau_1 + \left[\frac{1+z}{3(1-p)z} \right]^{\frac{1}{2}} \right\}. \quad (13)$$

The quantity τ_1 in this formula, that is, the boundary between the two regions mentioned above, is determined by Eq. (12). By means of the first of Eqs. (5) we find from this relation:

$$\tau_1 \left[\frac{3(1-p)z}{1+z} \right]^{\frac{1}{2}} = -\ln W \times \frac{\alpha_{12}}{\alpha_{2k}} \frac{g_1}{g_2} e^{h\nu_{12}/kT_*} p \left[\frac{z(1+z)}{3(1-p)} \right]^{\frac{1}{2}}. \quad (14)$$

Inserting Eq. (14) in Eq. (13) we finally obtain

$$E_{k2} = \left\{ -\ln W \times \frac{\alpha_{12}}{\alpha_{2k}} \frac{g_1}{g_2} e^{h\nu_{12}/kT_*} p \left[\frac{z(1+z)}{3(1-p)} \right]^{\frac{1}{2}} + 1 \right\} \\ \times \pi r_*^2 \frac{A_{k2} h \nu_{2k}}{\alpha_{1c}} \cdot \frac{n_k}{n_2} \cdot \frac{\alpha_{12}}{\alpha_{2k}} \left[\frac{1+z}{3(1-p)z} \right]^{\frac{1}{2}}. \tag{15}$$

This is the formula which we shall use in computing the total energy radiated by the envelope in the given line.

We shall now compare among themselves the quantities E_v^* , E_v , and E_{k2} , which we have just determined. For the characteristic ratio of the energy in the line to the energy radiated in the envelope in the continuous spectrum, we use the Zanstra quantities A_k , namely,

$$A_k = \frac{E_{k2}}{\nu_{2k} E_v}. \tag{16}$$

By means of the above formulae (1), (9), and (15), we compose Tables 14 and 15, which contain E_v/E_v^* and A_β in the region of H

Table 14. Values of the energy and of the quantity A_β for different values of x ($T_* = 20,000^\circ$, $T_e = 20,000^\circ$)

x	0.01	0.001	0.0001
E_v/E_v^*	0.03	0.24	2.1
A_β	0.42	0.030	0.002
$\tau_1 \sqrt{[3(1-p)z/(1+z)]}$	10	14	17
τ_{2c}^0	0.06	0.6	6.0

Table 15. Values of the energy and of the quantity A_β for different values of x ($T_* = 50,000^\circ$, $T_e = 50,000^\circ$)

x	1	0.1	0.01
E_v/E_v^*	0.6	1.6	11.5
A_β	0.35	0.045	0.002
$\tau_1 \sqrt{[3(1-p)z/(1+z)]}$	7	7	7
τ_{2c}^0	0.16	1.6	16

for different values of x . The first table refers to the case of $T_* = 20,000^\circ$, $T_e = 20,000^\circ$; the second to $T_* = 50,000^\circ$, $T_e = 20,000^\circ$. The necessary values of n_4/n_2 are taken from Tables 2 and 3 of Chapter I.

For W , the value 10^{-4} was adopted. The fourth line in each table gives the ratio of energy radiated in $H\beta$ in the nontransparent part of the envelope to the energy radiated in the same line in the transparent part. In the fifth line are given the optical depth of the envelope beyond the limit of the Balmer series (see Sec. 2). This quantity was computed by means of the formula

$$\tau_{2c}^0 = \alpha_{2c} \int_{r_1}^{r_2} n_2 dr = \frac{\alpha_{2c} g_2}{\alpha_{1c} g_1} e^{-h\nu_{12}/kT_*} \frac{B_{2c} \rho_{2c}^*}{x A_{21}}. \quad (17)$$

From these tables we can draw the following conclusions. With a decrease of x , the quantity E_ν/E_ν^* increases, and A_β decreases. With a sufficiently small value of x , the continuous spectrum of the envelope becomes brighter than the continuous spectrum of the star, and the quantity A_β comes near to the value observed in stellar spectra (of the order of 0.001).

From general physical principles it follows that the radiation of the envelope in the continuous spectrum must correspond to a temperature which is lower than that of the star. This is seen from Eq. (9), which gives the energy distribution in the continuous spectrum of the envelope. We find, for example, the color temperature of the envelope in the visual part of the spectrum. It follows from Eq. (9) that

$$E_\nu \sim e^{-h\nu/kT_e}. \quad (18)$$

Usually we assume that the energy distribution in the continuous spectrum is given by the Planck formula for a certain temperature T' .

$$E_\nu \sim \frac{\nu^3}{e^{-h\nu/kT'} - 1}. \quad (19)$$

It is easy to see that the temperatures T' and T_e are connected by

$$\frac{h\nu}{k} \left(\frac{1}{T'} - \frac{1}{T_e} \right) = 3. \quad (20)$$

In the region of $H\beta$, we have instead of Eq. (20)

$$10,000 \left(\frac{1}{T'} - \frac{1}{T_e} \right) = 1. \quad (21)$$

This means that T' is much smaller than T_e . Thus, for example, with $T_e = 20,000^\circ$, we obtain $T' = 6700^\circ$. Therefore the radiation of the envelope, even for very high temperature T_e , will appear of low temperature.

We remark that the distribution of energy in stellar spectra of later classes with bright lines is poorly determined, owing to the complication of the absorption spectrum. For the star Z Andromedae, Plaskett has obtained $T' = 5200^\circ \pm 900^\circ$. No doubt, however, the color temperature of the long-period variables is considerably lower than this, because these stars are very "red."

In explanation of the strong reddening of the envelope in comparison with the star, one should keep in mind that the foregoing results are not quite correct. It is a question of the derivation of Eq. (5), which determines the excitation and ionization in the envelope, in which we assumed that the optical depth of the envelope beyond the limits of the subordinate series is less than unity. In reality, as is seen from Tables 14 and 15, the quantity τ_{2c}^0 increases with decreasing x and becomes larger than unity for sufficiently small values of x . In the latter case, the theory of radiative equilibrium becomes very complicated, and a detailed investigation of this case is beyond the frame of the present work. Nevertheless, the following is quite obvious. If $\tau_{1c}^0 \gg 1$, but $\tau_{2c}^0 < 1$, the envelope absorbs the whole radiation of the star beyond the limit of the fundamental series, but only part of the radiation of the star beyond the limits of subordinate series, and it radiates energy beyond the limits of subordinate series and in spectral lines. And yet the quanta beyond the limit of the fundamental series which are radiated by the envelope are reabsorbed in the envelope, and this continues until almost all the quanta will be transformed into quanta of lower frequency. This case has been discussed above, and has led to Eq. (9), which determines the radiation of the envelope in the continuous spectrum. If, however, not only $\tau_{1c}^0 \gg 1$, but also $\tau_{2c}^0 > 1$, the envelope transforms not only the radiation beyond the limits of the

fundamental series, but also the radiation beyond the limits of the subordinate series, into radiation of lower frequency. It is obvious that this refers to the radiation coming from the star as well as the radiation peculiar to the envelope. It is clear that the radiation of the envelope in the second case will correspond to an even lower temperature than in the first case. Such is, in its general features, the origin of the continuous spectrum of late type radiated by the envelope.

As has been explained above, the appearance of the spectrum of the envelope depends fundamentally on x , which is given by

$$x = \frac{1}{3W} \cdot \frac{1}{2\mu} \cdot \frac{dv}{n_1 \alpha_{12} dr}. \quad (22)$$

According to the ionization formula, $Wn_1 \sim n_e n^+$. This means that the greater the density of the envelope, the smaller the parameter x . Therefore we reach the conclusion that, other things being equal (that is, for identical values of the temperature of the central star, dilution coefficient, and so forth), the continuous spectrum of the envelope will be later, the greater the density of the envelope.

It remains still to explain the origin of the absorption spectra of later type. For this we turn to the situation in the outer part of the envelope. With a sufficiently small value of x , the interior of the envelope completely occults the high-frequency radiation coming from the star. Therefore the external part of the envelope finds itself under the effect of the low-temperature radiation of the envelope itself. Consequently, in this part the excitation and ionization of atoms must correspond to a considerably lower temperature. In other words, here there must exist neutral atoms of metals and molecular compounds. Thus the absorption spectrum of late type is produced in the outer part of the envelope.

In this way we come to the following conclusion. The internal part of the envelope plays the part of a photosphere, producing the continuous spectrum of late type. In the external part of the envelope, which plays the part of an atmosphere, there occurs absorption lines of neutral metals and molecular bands. In the layer which is directly adjoining the photosphere occur emission lines of hydrogen and ionized metals. At sufficiently high temperature of the star, there

must also be observed emission lines of atoms of very high ionization potential. We should emphasize that in our results it was essential that the optical depth of the envelope, beyond the limits of the subordinate series, should be greater than unity. Therefore we must investigate in detail the causes that influence the quantity τ_{2c}^0 . We devote the following section to this question.

2. Optical Depth of the Envelope Beyond the Limits of Subordinate Series

In the preceding chapter, three causes were indicated which diminish the degree of excitation and ionization in the envelope: (1) the Doppler effect; (2) collisions of the second kind; (3) presence of general absorption. In deriving Eq. (5) we took into account only the Doppler effect. We shall now consider the other two.

The third of these causes seems at the first glance to be especially "dangerous." We should, however, keep in mind that, in the case considered, the general absorption in the envelope is nothing less than the absorption beyond the limit of the subordinate series of a given atom, and this absorption decreases with a decrease in the degree of excitation. Consequently, in this case the degree of excitation and ionization must fall more slowly than was derived in the preceding chapter (Sec. 3*b*). We again assume that the atoms possess only three states, and shall begin from the Eqs. (IV.62). Neglecting a very small quantity η_1 and taking collisions into account, instead of this equation we have:

$$\begin{aligned}\frac{d^2 \bar{K}_{13}}{d\tau^2} &= 3(1 - p + \eta) \bar{K}_{13} - \frac{\gamma}{q} \bar{K}_{12}, \\ q^2 \frac{d^2 \bar{K}_{12}}{d\tau^2} &= (\beta + \gamma + \delta) \bar{K}_{12} - 3q(1 - p) \bar{K}_{13}.\end{aligned}\tag{23}$$

From this we find, for the part of the envelope sufficiently remote from the boundaries,

$$\frac{d^2 \bar{K}_{13}}{d\tau^2} = 3 \left[\eta + (1 - p) \frac{\beta + \delta}{\beta + \gamma + \delta} \right] \bar{K}_{13}.\tag{24}$$

The quantity η which enters this equation is determined by means of the first of Eqs. (IV.61). Denoting by μ the ratio of the coefficient

of absorption beyond the limit of the fundamental series, due to transitions of type $2 \rightarrow 3$, to the coefficient of absorption beyond the limits of the subordinate series ($\mu = \alpha_{13}/\alpha_{23}$), and having in view the relation

$$\frac{\alpha_{23}}{\alpha_{13}} = \frac{1-p}{p} \frac{\bar{K}_{13}}{\bar{K}_{23}}, \quad (25)$$

we find the quantity

$$\eta = \mu \frac{1-p}{p} \frac{\bar{K}_{13}}{\bar{K}_{23}}. \quad (26)$$

But $\bar{K}_{23} = \frac{1}{4}S_{23}$. Therefore, inserting Eq. (26) in Eq. (24) we obtain

$$\frac{d^2 \bar{K}_{13}}{d\tau^2} = 3 \left[4\mu \frac{1-p}{pS_{23}} \bar{K}_{13} + (1-p) \frac{\beta + \delta}{\beta + \gamma + \delta} \right] \bar{K}_{13}. \quad (27)$$

The solution of Eq. (27) is in the form of an elliptic integral:

$$\int \frac{d\bar{K}_{13}}{\left[8\mu \frac{1-p}{pS_{23}} \bar{K}_{13}^3 + 3(1-p) \frac{\beta + \delta}{\beta + \gamma + \delta} \bar{K}_{13}^2 + C_1 \right]^{\frac{1}{2}}} = \tau + C_2, \quad (28)$$

where C_1 and C_2 are arbitrary constants. These should be determined from the boundary conditions:

$$-\frac{1}{3} \frac{d\bar{K}_{13}}{d\tau} = \frac{1}{4} S_{13}(\tau = 0); \quad -\frac{2}{3} \frac{d\bar{K}_{13}}{d\tau} = \bar{K}_{13}(\tau = \tau_0). \quad (29)$$

If the total absorption in the envelope plays a greater part than Doppler effect and collisions, then the solution of Eq. (27) with the boundary conditions (29) for the case $\tau_0 = \infty$ has the form

$$\bar{K}_{13} = \frac{\frac{3}{4}S_{13}}{\mu\tau_{23}^0(1 + \frac{1}{2}\mu\tau_{23}^0\tau)}, \quad (30)$$

where τ_{23}^0 is the optical depth of the envelope beyond the limit of the subordinate series. This quantity is

$$\tau_{23}^0 = \left(6 \frac{1-p}{p\mu^2} \cdot \frac{S_{13}}{S_{23}} \right)^{\frac{1}{2}}. \quad (31)$$

In the opposite case, the solution of Eq. (27) is obtained in the form derived by us earlier (see the first formula IV.32), namely,

$$\bar{K}_{13} = \frac{3S_{13}}{4\lambda} e^{-\lambda\tau}, \quad (32)$$

where

$$\lambda = \left[3(1-p) \frac{\beta + \delta}{\beta + \gamma + \delta} \right]^{\frac{1}{2}} \quad (33)$$

For the quantity τ_{23}^0 in this case we find:

$$\tau_{23}^0 = 3 \frac{1-p}{p\lambda^2} \frac{S_{13}}{S_{23}}. \quad (34)$$

We shall now estimate the expressions for τ_{23}^0 . For the hydrogen atom, $p = \frac{1}{2}$, $\mu = 1/64$. Therefore for T_* of the order $30,000^\circ$, formula (31) gives a value of the order of several tens for τ_{23}^0 . For $W = 10^{-4}$, $n_e = 10^{11}$, and $\beta < 10^{-6}$, Eq. (34) gives for τ_{23}^0 a quantity of the same order. We see in consequence that taking into account the collisions and general absorption does not appreciably decrease the optical depth of the envelope beyond the limit of the subordinate series.

The solution given above appears, however, to be not quite accurate, because for $\tau_{23}^0 > 1$ the quantity \bar{K}_{23} cannot be taken as constant. However, it is easy to see that for frequency ν_{23} we have essentially pure scattering. This means that \bar{K}_{23} cannot change greatly in the envelope. Therefore even an accurate solution of the problem cannot lead to a result that differs appreciably from those obtained above. We shall now give an accurate solution of the problem, neglecting collisions and Doppler effect for the sake of simplicity.

Instead of the one equation (27) we now have the following system of three equations:

$$\frac{d^2 \bar{K}_{13}}{d\tau^2} = 3\mu \frac{1-p}{p} \frac{\bar{K}_{13}^2}{\bar{K}_{23}}; \quad \frac{d^2 \bar{K}_{23}}{d\tau^2} = 0; \quad \frac{d\tau_{23}}{d\tau} = \frac{1-p}{p} \frac{\bar{K}_{13}}{\bar{K}_{23}}. \quad (35)$$

The second of these, for boundary conditions analogous to Eqs. (29), gives

$$\bar{K}_{23} = \frac{3}{4} S_{23} (\tau_{23}^0 - \tau_{23}), \quad (36)$$

and from the third we find

$$\bar{K}_{13} = - \frac{4p}{3(1-p)} \frac{\bar{K}_{23}}{S_{23}} \frac{d\bar{K}_{23}}{d\tau}. \quad (37)$$

Inserting Eq. (37) in the first of Eqs. (35), we have

$$\frac{d^2}{d\tau^2} \left(\bar{K}_{23} \frac{d\bar{K}_{23}}{d\tau} \right) = - \frac{4\mu}{S_{23}} \bar{K}_{23} \left(\frac{d\bar{K}_{23}}{d\tau} \right)^2. \quad (38)$$

This apparently complicated equation has the following simple solution:

$$\bar{K}_{23} = \frac{\frac{3}{4} S_{23}^0 \tau_{23}^0}{1 + \frac{1}{4} \mu \tau_{23}^0 \tau}, \quad (39)$$

which satisfies the same boundary conditions (because for $\tau = 0$, τ_{23} must be $\tau = 0$, and for $\tau = \infty$, $\tau_{23} = \tau_{23}^0$).

Inserting Eq. (39) in Eq. (37), and using condition (29), we finally find for the constant τ_{23}^0 :

$$\bar{K}_{13} = \frac{S_{13}}{\mu \tau_{23}^0} \cdot \frac{1}{(1 + \frac{1}{4} \mu \tau_{23}^0 \tau)^3}, \quad (40)$$

$$\tau_{23}^0 = \left(\frac{16}{3} \frac{1-p}{p\mu^2} \frac{S_{13}}{S_{23}} \right)^{\frac{1}{4}}. \quad (41)$$

We see that in reality the accurate solution of the problem which is given by Eqs. (40) and (41) differs very little from the earlier solutions (30) and (31).

In conclusion we should mention that only factors have been taken into account above that produce a decreasing effect on the degree of excitation and ionization in the envelope. There exist factors, however, that work in the opposite direction (for example, collisions of the first kind). Therefore the estimates of the quantity τ_{23}^0 given above are in reality only minimal values.

3. General Considerations

In the preceding sections the following model was considered: a hot star, surrounded by an envelope which is located at some distance from the star. This model is characterized by three fundamental parameters: temperature of the star, coefficient of dilution,

and density of the envelope. It has been established above that by a suitable choice of these three parameters one can get any combination spectrum. From this follows two conclusions.

(1) All objects with emission lines in their spectra (gaseous nebulae, WR, P Cygni, and Be stars, novae and novalike stars, long-period variables, and so forth) differ from one another only in the difference in the values of the parameters mentioned. If, for example, the temperature of the star is sufficiently high, and the coefficient of dilution sufficiently small, with increasing density of the envelope there must appear a continuous spectrum and an absorption spectrum of increasingly later type, while the character of the emission lines remains unchanged. Thus all the objects enumerated line themselves up in the general theoretical scheme and apparently seem to be related according to their physical nature.

(2) For the explanation of changes in the spectrum of an individual object, it is sufficient to allow for small variations of the parameters mentioned. For example, with variation of the temperature in the interval from $15,000^{\circ}$ to $20,000^{\circ}$, the brightness of the star should change by a factor of 2 or 3, and bright hydrogen lines should either appear or disappear. Exactly such changes are characteristic of long-period variables. If the changes of the above-mentioned parameters are considerable, then an object of one kind should go over into an object of another kind.

In order to pronounce a judgment on the correctness of our conclusions, we must turn to the observations. It seems to us that these data provide strong witness in favor of the point of view enunciated.

Let us consider first the spectra of long-period variables. The fundamental question that interests us is whether the bright lines actually originate as a result of fluorescence. As is well known, the Balmer decrement in spectra of type Me is anomalous. However, this is not caused by an unknown excitation mechanism, but is explained by the occultation of the Balmer emissions by bands of TiO. In spectra of types Se and Ne the bands of TiO are absent, and the Balmer decrement is quite normal. (For details, see Chapter I, Sec. 5.) Yet in general it has been noticed that the emission spectrum of the long-period variable near maximum brightness, that

is, when the occultation by molecular bands is minimal, is similar to the emission spectrum of stars of type Be and new stars at the moment of the appearance of the bright lines. This fact alone indicates that the emission spectrum of long-period variables is apparently excited by the same mechanism as the emission spectra of Be stars and novae, that is, by photoionization and recombination. To this we must add that in spectra of novae there has also been noted some discrepancy between the emission spectrum which corresponds to B class, and the absorption spectrum, which is usually for Classes A and F. In addition, in spectra of some novae (for example, Nova Herculis 1934), side by side with bright lines are observed the absorption bands of molecular compounds. Consequently, spectra of novae in this period appear to have some similarity with the spectra of stars of later class with bright lines.

From observation it follows also that the bright lines in the spectra of long-period variables originate in deeper layers than the absorption lines. This fact is in complete agreement with the result obtained in Sec. 1 of the present chapter.

We turn our attention to the fact that in the present chapter, as in all this work, we assume that the envelopes are in motion. This assumption is undoubtedly correct in reference to the long-period variables. In general, as Shajn⁴ remarks, emission is always connected with motion. There is interest in a detailed study of the characteristic motions of the upper layers of long-period variables. Regrettably, the observational data on this point are very meager. For a single one of these stars, Mira Ceti, Joy⁵ has determined the displacement of the spectral lines for the whole cycle. He has established that the radial velocity curves for the bright lines and the dark lines are strongly displaced relative to each other. At the moment of maximum brightness, the curve of emission lines has a minimum and the curve of absorption lines a maximum. In reference to the former, one can say in general that it is similar to a mirror image of the light curve. In reference to other long-period variables we must be satisfied with statistical data. These data are: (1) the difference in radial velocities for bright and dark lines is always negative ($v_e - v_a < 0$); (2) the K effect, as determined for the bright lines, is about -15 km/sec; (3) the K effect as determined for the

dark lines is about zero (see the work of Merrill⁶ and Allen⁷). From these data it follows that the layer in which the bright lines are formed is moving in the direction of the observer. In other words, there occurs an ejection of material from the long-period variables. The first such conclusion, as far as we know, was made by Shajn.⁴ Some doubts are suggested by the fact that $K_a \cong 0$. This result, however, is very uncertain, because the velocity v_a is not determined directly, but with reference to v_e near maximum brightness. In that time the displacements of absorption lines appear to be algebraically small (see the work of Joy, quoted above). Therefore the value $K_a \cong 0$, just given, should be taken as near to the upper limit of the K term for the absorption lines. If this is so, then the hypothesis of the ejection of matter from long-period variables appears to be quite probable. Here the process of ejection should not be stationary, but of variable character. The ejected matter itself should suffer considerable retardation.

Of especial interest are the objects with emission lines in their spectra, which undergo rapid transformation from one class into another. Such transformations bear convincing witness in favor of the view that all objects with bright lines are related. We shall give a few examples of such objects.

(1) T Coronae Borealis. This star flared up as a nova in 1866. After that it became a giant M star with emission lines. In 1946 it again flared up as a nova.⁸

(2) Z Andromedae. This star, with a typical combination spectrum (the later spectrum of type M, the earlier of type W) suddenly became a P Cygni star in 1933. After some time the star returned to its usual state.

(3) R Aquarii. The spectrum of this star is that of a typical long-period variable. However, from time to time there appears a superimposed high-temperature spectrum, with bright lines of hydrogen, helium, and other elements.

According to Berman, all these stars seem to be double stars, consisting of a cool giant and a hot component. For the star R Aquarii, Berman³ gave a detailed interpretation of this kind. Here the chief part of the change of brightness is ascribed to the hypothetical component. Other stars of this group are found to be in the

same situation. On attentive consideration, this hypothesis of duplicity seems generally to be quite artificial. However, on our point of view, the stars of Z Andromedae type are single, hot stars surrounded by envelopes which give spectra of later type. Here the brightness of the continuous spectrum of the envelope is comparable with the brightness of the continuous spectrum of the star. In consequence, two superimposed spectra are observed. In accordance with our interpretation, the red end of the continuous spectrum belongs to the envelope, and the violet end, to the star. The oscillation of brightness of the continuous spectrum of the envelope can explain the appearance and disappearance of the spectrum of the star.

In concluding this chapter, we may remark that observation in general confirms the theoretical considerations given above. There is no doubt that these considerations are in need of future development and refinement. In particular it is not quite clear whether the model given above has reality, or is only a first approximation to a star with an extended atmosphere. For the discussion of this and many other questions, more observational data are needed.

SUMMARY

This investigation is devoted to the most interesting objects in the sky. In it there are subsequently considered stars of early spectral classes with bright lines (stars of WR, P Cygni, and Be types), planetary nebulae, new stars, and stars of later classes with bright lines (long-period variables, stars of type Z Andromedae, and so forth). The presence of bright lines in their spectra appears to be a superficial characteristic connecting all these stars into one group. In reality, however, there exists a more fundamental connection between these objects. They have the following general characteristics.

(1) Intense ejection of material, leading to formation of moving envelopes. Envelopes transform high-frequency radiation from the stars into radiation of lower frequency.

(2) The result of such fluorescence is the occurrence of bright lines in the stellar spectrum.

(3) Envelopes are nontransparent for radiation of lines in the subordinate series, except in the nebulae, which are nontransparent only for the radiation of lines of the fundamental series. In consequence of this, the process of transformation of energy in the envelopes appears to be very complicated.

For detailed study of phenomena occurring in the ejected envelopes we should build up a new theory of radiative equilibrium which takes into account motions of the envelope—the Doppler effect in line radiation within lines. In the present work we have given the basis for such a theory and we have shown that for the moving envelopes the theory of radiative equilibrium can be much simpler than for stationary envelopes. Applications with chief attention to confirming the correctness of the above-mentioned characteristics for all stars were considered.

We may summarize the important results as follows.

(1) For the moving medium we have found the degree of excitation and ionization as functions of T and the parameter x , which is the ratio of the velocity gradient to the coefficient of dilution. We

establish that in a moving envelope the degree of excitation is much less than the Boltzmann one for a given temperature (Chapter I, Secs. 1 and 2).

(2) We have computed relative intensities of emission lines, particularly the Balmer decrement, as functions of T and x . It is shown that theoretical intensities are in good agreement with observed ones (Chapter I, Secs. 3 and 5).

(3) For the stars of early spectral classes with bright lines, we have explained how the degree of excitation changes, also the radiative ability within the line, along the radius. At the same time we discover stratification of radiation in the atmosphere. We have given the formulae which determine the lower and upper boundaries of the reversing layer. It is established that the stars investigated possess very extensive reversing layers and that the major part of radiation within the line falls in part of the reversing layer and not on the outside transparent envelope, as has been usually considered (Chapter II, Secs. 1 and 2).

(4) On the basis of this new point of view, we have given the theory of the contours of spectral lines formed in moving atmospheres. The contours were computed for (a) the expanding atmosphere and (b) the expanding and rotating atmosphere. It is shown that the first contours are in close agreement for the WR, P Cygni, and new stars, and the second type for the Be stars (Chapter II, Sec. 3).

(5) Formulae are given for the ionization (determining the degree of ionization at different optical depths) for planetary nebulae (Chapter III, Sec. 2) and for moving envelopes of small radius (Chapter IV, Secs. 1 and 2).

(6) We determine the density of L_α radiation in the nebula and the amount of radiation pressure caused by this radiation. It is established that, owing to the Doppler effect, the radiation pressure in the nebula plays a much smaller part than was believed before (Chapter III, Sec. 2). Analogous computations were made for envelopes of small radius (Chapter IV, Sec. 4).

(7) A method is given for the determination of temperature of the envelopes. The application of this method for planetary nebulae leads to temperatures of 8000° – $14,000^\circ$ (Chapter III, Sec. 3). It is

shown that the temperature of the envelopes of small radius may also be much lower than the temperature of the stars (Chapter IV, Sec. 4).

(8) A refinement of the Zanstra method is given: (*a*) taking into account the optical depth of the envelope beyond the limit of the fundamental series (Chapter II, Sec. 1); (*b*) for estimating the number of quanta absorbed by the envelope beyond the limit of the subordinate series (Chapter IV, Sec. 4); and (*c*) taking into account the amount of energy of free electrons which is expended for the excitation of the "nebulium" line (Chapter III, Sec. 3).

(9) It is shown that for sufficiently small values of the moving envelope, together with the emission lines, should give a continuous absorption spectrum of later type than the star. Thus we give the chief explanation for the origin of the bright lines in the spectra of late class. We have given some conclusions on the physical relations of all these bright line objects (Chapter V, Secs. 1–3).

It is self-evident that the foregoing results do not exhaust the possibilities of the theory. The theory itself must be refined and supplemented. The fact that for moving envelopes the theory of radiative equilibrium seems to be much simpler than for the stationary atmospheres gives the basis for hope that in the future the theory will become one of the best developed in astrophysics.

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INDEX

- Absorption, ix, 5, 26, 28, 30, 33, 35, 36, 37, 38, 42, 45, 46, 72, 74, 90
- Allen, C., 97
- Aller, G., 41, 44
- Ambartsumian, V. A., x, 8, 41, 49, 50, 58, 62, 70, 76
- Atmosphere, stellar, viii, x, xi, 21, 26, 29, 31, 99, 100

- Baker, T., 41, 44
- Baldwin, R., 29
- Balmer decrement, x, 9–11, 13–18, 58, 95, 100
- Balmer series, 13–18, 23, 28, 59–60, 85, 88
- Beals, C., 15, 16, 17, 24, 32, 35
- Berman, C., 57, 59, 83, 97
- Be stars, vii, x, 15, 20, 21, 24, 30, 36, 37, 39, 95, 96, 99, 100
- Boltzman constant, 7, 52
- Boltzman formula, x, 9, 72, 75, 100
- Bright lines, vii, ix, xi, xii, 14, 18, 19, 20, 30, 36, 39, 82, 83, 84

- Carroll, J., 31
- “Cascade,” ix
- Chandrasekhar, S., 32, 41
- Cillié, G., x, 7, 15, 16, 18, 41, 51, 54, 59, 60
- Combination spectra, 84

- Dilution coefficient, 5, 9, 56, 69, 90, 95, 99
- Doppler effect, x, xi, 4, 5, 23, 31, 32, 50, 63, 65, 70, 85, 91, 92, 93, 99, 100

- Eddington, A. S., ix, 44, 66
- Einstein coefficient of absorption, 4, 12, 63
- Ejection of matter, vii, ix, 20, 99
- Emission lines, x, xi, 5, 17, 30, 32, 36, 37, 40, 83, 90

- Envelopes, vii, ix, x, xi, 14, 17, 18, 21, 30, 36, 37, 41, 62, 63, 68, 75, 81, 82, 84, 95, 98, 99
- Excitation, degree of, ix, xi, 4, 6, 20, 21, 26, 27, 28, 62, 68, 72, 74, 85

- Fluorescence, viii, ix, 81, 83, 99
- Fundamental series, xi, 3, 7, 25, 26, 41, 62, 71, 76, 78, 79, 80, 82, 86, 89, 90, 92, 99

- Gaposchkin, S., 25
- Gaseous medium, viii, 3, 7, 37, 41, 75, 84
- Gerasimovich, B., 31, 32
- Goedicke, G., 29
- Gordeladze, V., 76
- Greaves, T., 15

- Hagihara, H., 41
- Helium, 4, 7, 9, 11, 20, 76
- Heney, L., 70
- Hydrogen, 4, 7, 8, 18, 20, 25, 29, 50, 57, 73, 76, 90, 93

- Ionization, degree of, ix, x, xi, 3, 6, 9, 41, 42, 43, 44, 51, 55, 62

- Joy, A., 96, 97

- Karpov, P., 15

- L_2 -radiation, ix, x, 41, 42, 44, 45, 46, 49, 58
- Lyman series, 22, 23, 59, 61

- Martin, E., 15
- McCrea, W., 32, 81
- McLaughlin, D., 37
- Melnikov, O., 32
- Menzel, D. H., 41, 44, 58, 61
- Merrill, P., 97
- Milne, E. A., ix, 17

- Mitra, 32
 Miyamoto, S., 54, 58, 59
 Mohler, O., 24
 Mustel, E., 36, 80, 81

 Nebular lines, 51
 "Nebulium," 56, 57, 58, 60
 Nova, vii, xi, 18, 30, 31, 35, 36, 40, 62, 81, 95, 99
 Nova Herculis, 15, 96
 Nova Lucerta, 18
 Novalike star, vii, 95

 Oehler, O., 60

 Page, T., 57, 60
 P Cygni stars, vii, 15, 20, 21, 24, 25, 30, 31, 35, 40, 95, 97, 99, 100
 Pickering series, 16
 Planck constant, 7, 61, 88
 Planetary nebulae, vii, ix, xi, 44, 49, 62, 66, 99, 100
 Plaskett, J. S., 89
 Popper, O., 18

 Radiative equilibrium, viii, ix, x, xi, 3, 31, 41, 63, 65, 74, 81, 100
 Radiation flux, viii, 41, 80
 R Aquarii, 83, 97
 Russell, H. N., 82

 Sayer, A., 76
 Schuster, A., 44, 81
 Schwarzschild, K., ix, 44
 Shajn, G., 18, 96, 97
 Stoy, R., 60, 61
 Struve, O., 37
 Subordinate series, ix, x, xi, 7, 18, 27, 41, 45, 62, 71, 78, 79, 80, 82
 Supernovae, vii, 49

 T Coronae Borealis, 97
 Temperature, ix, x, xi, 6, 8, 15, 24, 25, 51-61, 76-79

 Velocity gradient, vii, x, xi, 3, 5, 6, 7, 19, 20, 26, 32, 41, 45, 48, 49, 69, 70, 73, 81, 90, 99
 Vorontsov-Velyaminov, B., 16, 32, 36

 Wilson, O., 32
 Wolf Rayet stars, vii, x, 15, 16, 17, 20, 21, 25, 30, 31, 35, 36, 40, 76, 95, 99, 100
 Wurm, C., 82

 Z Andromedae, 82, 89, 97, 98, 99
 Zanstra, H., x, xi, 24, 25, 41, 49, 50, 56, 57, 60, 61, 76, 77, 87, 100